AN ELEMENTARY 3D VISUALIZATION LEARNING TRAJECTORY

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This research team has developed a learning trajectory on 3D visualization for elementary children. This paper focuses on the interchange among the three Spatial Operational Capacity framework representations, namely, 3D models, 2D conventional diagrams, and semiotic abstract representations, and the critical role of a dynamic computer interface that simulates the representations. The trajectory is described through an actions-on-objects (Connell, 2001) lens and the project's strong problem-solving approaches that are critical to its successful enactment.

The National Research Council's report, *Learning to Think Spatially* (2006), identifies spatial thinking as a significant gap in the K-12 curriculum, which, they claim, is presumed throughout but is formally and systematically taught nowhere. They believe that spatial thinking *is* the start of successful thinking and problem solving, an integral part of mathematical and scientific literacy. The importance of visual processing has been documented by researchers who have examined students' performance in higher-level mathematics. For example, Tall et al (2001) found that to be successful in abstract axiomatic mathematics, students should be proficient in both symbolic and visual cognition; Dreyfus (1991) calls for integration across algebraic, visual and verbal abilities; and, Presmeg (1992) believes that imagistic processing is an essential component in one's development of abstraction and generalization.

The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (NCTM, 2000) recommends that in their early years of schooling, students should develop visualization skills through hands-on experiences with a variety of geometric objects and use technology to dynamically transform simulations of two- and three-dimensional objects. Later, they should analyze and draw perspective views, count component parts, and describe attributes that cannot be seen but can be inferred. Students need to learn to physically and mentally transform objects in systematic ways as they develop spatial knowledge.

Using design-research (Cobb, et al, 2003) principles this research team has developed a learning trajectory on spatial development for elementary children guided by the Spatial Operational Capacity (SOC) framework developed by van Niekerk (1997) based on Yakimanskaya's (1991) work. This paper focuses on the interchange among SOC representations and the critical role of a dynamic computer interface, through an actions-on-objects (Connell, 2001) lens. The project's strong problem-solving approaches make this possible. It is conducted in a dual-language urban elementary school within one of the largest public school districts in the mid-southwestern United States. More than 70% of its students are designated "At Risk" and at least 50% of its students are English Language Learners.

Theoretical Frameworks

The *spatial operation capacity* (SOC) framework (van Niekerk, 1997; Sack & van Niekerk, 2009) that guides this study exposes children to activities that require them to act on a variety of physical and mental objects and transformations, as prescribed by the National Council of Teachers of Mathematics (NCTM, 2000) to develop the skills necessary for solving spatial problems. The framework (see Figure 1) uses: full-scale figures, that, in this study, are created from loose cubes or Soma figures, made from 27 unit cubes glued together in different 3- or 4-

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cube arrangements (see Figure 2); conventional 2D pictures that resemble the 3D figures; semiotic representations such as front, top and side views or numeric top-view codings that do not obviously resemble the 3D figures; and, verbal descriptions that may be accompanied by gestures using appropriate mathematical language (Sack & Vazquez, 2008).

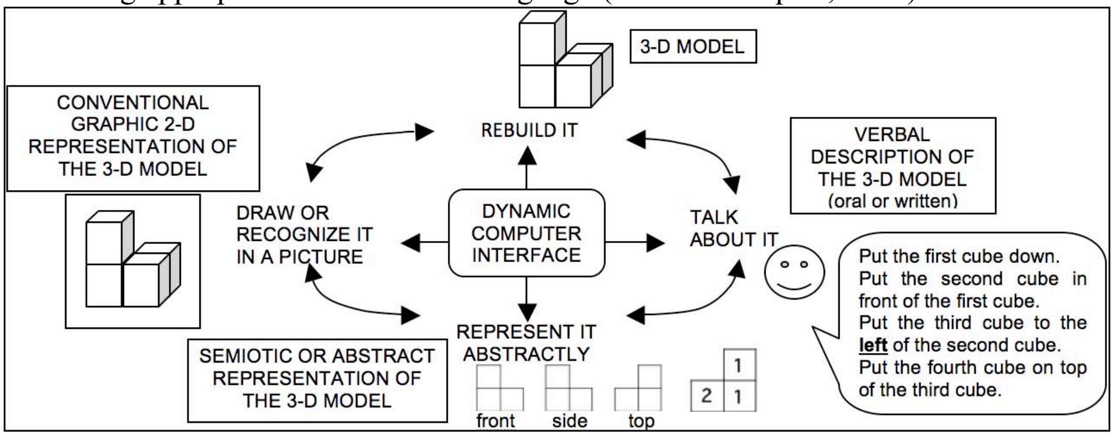

Figure 1. Multiple representations within 3-D visualization

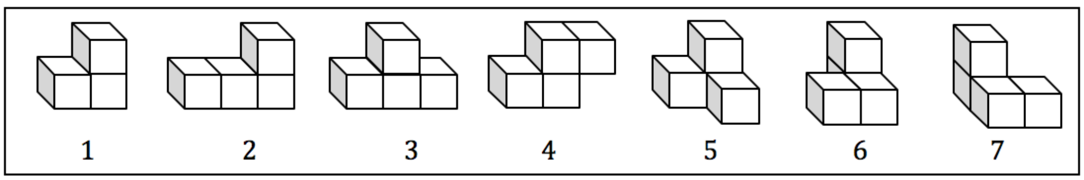

Figure 2. The Soma set can be made by gluing unit cubes together.

The project utilizes a dynamic computer interface, Geocadabra (Lecluse, 2005). Through its Construction Box module, complex, multi-cube structures can be viewed as 2-D conventional representations or as top, side and front views or numeric top-view grid codings (see Figure 3). These options can be (de)selected according to instructional goals. The Control-line-of-view option allows the user to move the figure dynamically using the mouse or by clicking on the arrows at the ends of the space's triaxial system.

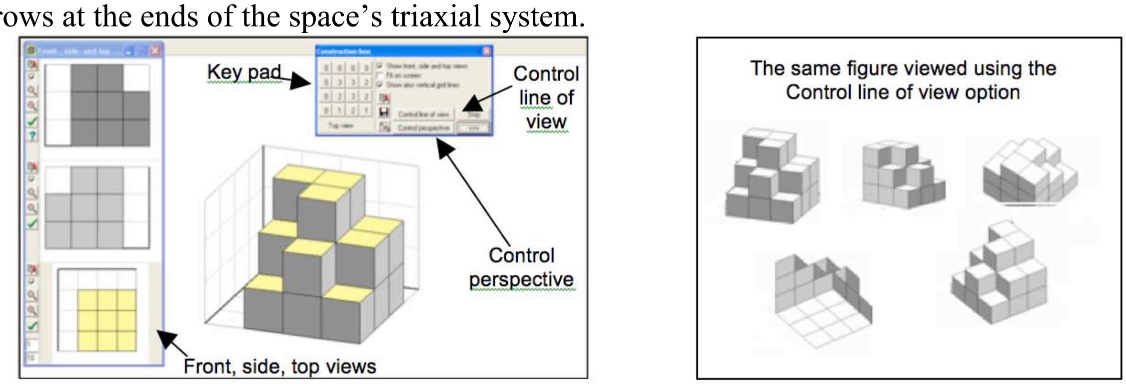

Figure 3. The Geocadabra Construction Box

Connell's (2001) *action on objects* metaphor guides the discussion section of this paper. Through carefully designed activities, children act strategically upon manipulative objects as they solve problems. Computer images that replicate the attributes of the physical objects then behave as real objects in the mind of the learner. The Geocadabra Construction Box interface integrates the SOC representations in the form of a dynamic image that can be moved to provide

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the same views as if moving about a 3-D object; a 2-D image when the figure remains static; and if selected, simultaneous semiotic top-view numeric, or face view representations. Follow up problems or questions require the child to relate to the newly instantiated and defined object of thought, which becomes the basis upon which later mathematical thinking occurs. This model extends as the child develops his or her own problems based upon the objects that were recently defined. "This ability, to pose one's own problems and to then successfully solve these problems, provides further opportunity for growth in mathematical thinking and problem solving" (p. 161).

Methodology and Context

Since the project's inception in 2007-2008, a university-based researcher and two teacher-researchers have formed the research team working with a group of children weekly for one hour in teacher-researcher, Vazquez' 3rd-grade classroom within the school's existing after-school program. English and Spanish parent/guardian and student consent-to-participate forms are sent home to parents of all 3rd grade children. All respondents are accepted into the program. The research team uses socially mediated instructional approaches to support a problem-solving environment that fosters students' creativity according to readiness and interest.

Design research methodology (Cobb, et al, 2003) guides this study's instructional decisions based on learning trajectories developed from an instrumentalist standpoint (Baroody, et al, 2004). This conceptual and problem-solving approach aims for "mastery of basic skills, conceptual learning, and mathematical thinking" using any "relatively efficient and effective procedure as opposed to a predetermined or standard one" (p. 228). Each lesson is part of a design experiment followed by a retrospective analysis in which the research team determines the actual outcomes and then plans the next lesson. This may be an iteration of the last lesson to improve the outcomes, a rejection of the last lesson if it failed to produce adequate progress toward the desired outcomes, or a change in direction if unexpected, but interesting, outcomes arose that are worthy of more attention. Data corpus consists of formal and informal interviews, video-recordings and transcriptions, field notes, student products and lesson notes.

Results and Discussion

During the learning trajectory's introductory lessons, children interact with loose cubes and the Soma figures initially solving problems with the 3D models and with 2D task cards (e.g., see Figure 4a). These illustrate a variety of assemblies of two Soma combinations in different orientations requiring figure identification and classification. Thus, learners become familiar with the SOC framework's 3D and 2D conventional graphic representations before they are introduced to the Geocadabra virtual interface. By the middle of the second month, children begin to digitally reproduce figures printed in a customized manual (e.g., see Figure 4b). These activities provide the children opportunities to coordinate numeric top-view codings with 2D pictures. In addition, there is a strong focus on enumeration of cubes in the manual's figures. Whereas beginning learners generally are able to determine the numbers of cubes in relatively simple figures (such as the left-most task card in Figure 4a), very few can do so with figures containing hidden cubes (as in Figure 4b). Battista (1999) has shown that many learners count visible faces when asked to find the number of unit cubes in 3D rectangular arrays, often double counting edge cubes and triple counting vertex cubes. This research team's pre-program interviews, September, 2010, using both 3D rectangular arrays and multi-level structures as in Figure 4b, yielded similar results. Some children recognize that cubes are hidden but lack the mental structuring capacity to determine precisely how many in a logical fashion.

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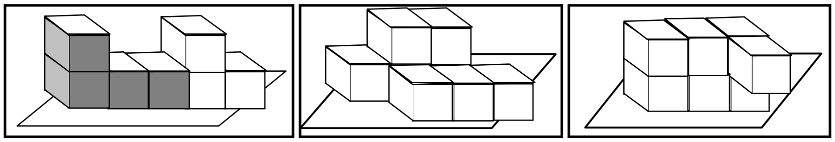


Figure 4a. Task cards increasing in level of difficulty

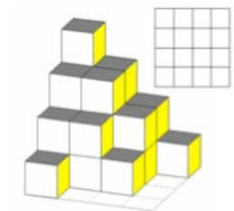


Figure 4b. Task 1*f*

By working systematically with increasingly more complex figures and by enabling the Geocadabra Construction Box's Control-line-of-view function has developed children's capacity to solve such enumeration problems. The learner's ability to see the digital figure in the same way as a 3D model enhances the digital experience. This way, since a stack of cubes grows by repetitive clicks on the corresponding position on the number grid, the children know that the computer figures can have no holes beneath any visible cubes. The ability to rotate the digital image allows them to see the sides, back and bottom views of the figure. Thus, movement from conventional 2D graphical images to virtual 3D models via the numeric top-view semiotic representation is realized.

After developing reasonable proficiency with the Geocadabra Construction Box through the manual's tasks, more open-ended problems are posed to further develop children's mental imaging capacity with respect to volume concepts. They are expected to correlate the numbers in the Geocadabra Construction Box grid with the height of each stack of cubes in each 2D picture; and, to verify that the number of cubes in the 2D picture is the same as the sum of the numbers in the Geocadabra Construction Box grid. Children create their own structures consisting of 24 unit cubes, using the Geocadabra Construction Box. Initially, they may choose to build the figure using 24 loose cubes, but most discard the loose cubes almost immediately. Conventional 2D images of these digital figures are converted by the researcher into new task cards essentially created by the children. The creator then draws the numeric top-view code and a peer decodes and re-creates it on the computer, first hiding the visual figure, and then showing it to check that it matches the task card figure. In addition, the peer checker ensures that the figure consists of exactly 24 cubes by adding the numbers in the top-view numeric grid on the computer. Examples of student-created task cards are shown in Figure 5. Enumeration of these figures using symmetry and slicing is encouraged and shared during whole class discussions when verbal language is developed. The research team has noted that children seem to have more difficulty enumerating 3D rectangular arrays than the types of figures shown in Figure 5. However, with practice, they are able to represent 3D rectangular arrays using numeric top-view coding grids. This is a critical step in their understanding of formula-based volume concepts (see Sack, & Vazquez, 2010).

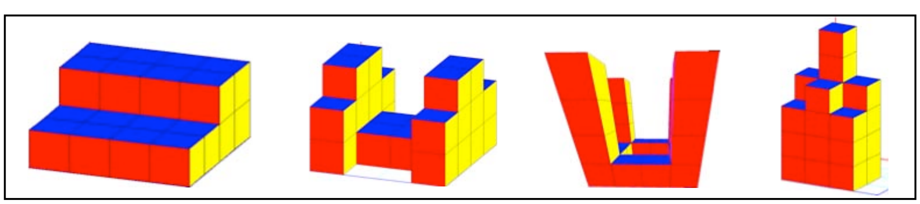
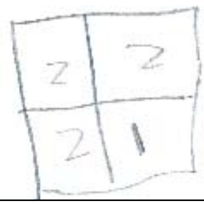
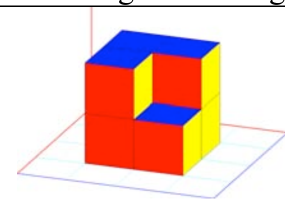
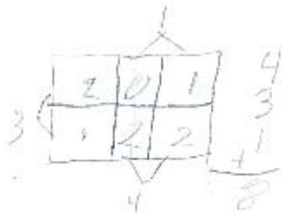
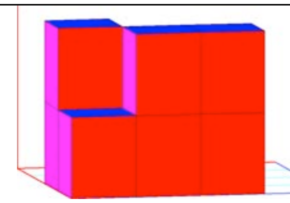
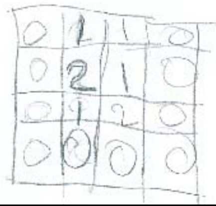
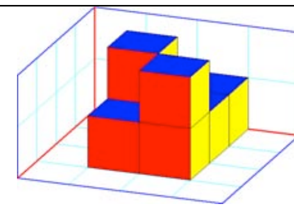
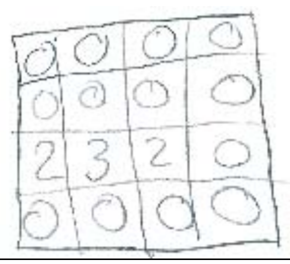
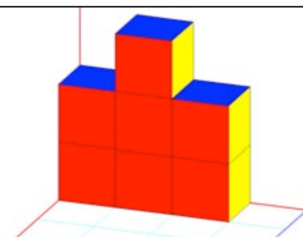


Figure 5. Students' 24-cube figures

Numeric top-view coding	Peer solutions	Original task figure
A 	Soma #1 and Soma #5 OR Soma #1 and Soma #6	
B 	Soma #6 and Soma #7	
C 	Soma #5 and Soma #6 OR Soma #2 and Soma #3	
D 	Soma #4 and Soma #1 OR Soma #2 and Soma #1	

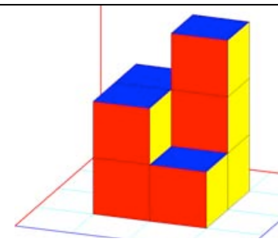
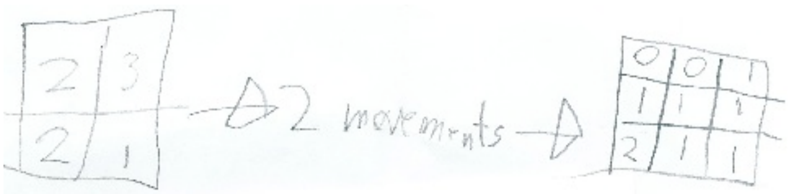
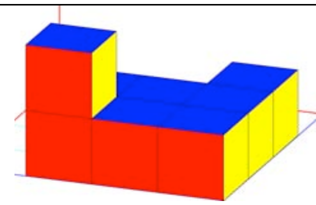
Numeric top-view coding	Peer solutions	Original task figure
E 	E1 Soma #7 and Soma #4 E2 Soma #3 and Soma #5 OR Soma #6 and Soma #2	
F 		

Table 1. Examples of student-made semiotic puzzles and peer solutions

The discussion now moves to Connell's (2001) notion of deepening mathematical understanding through self-developed problems that encompass all of the visual SOC representations. Using a complete set of seven Soma figures, children first select two to create a 3D assembly structure that can be reproduced on the Geocadabra Construction Box, unlike the rightmost structure in Figure 4a. The following week, without the aid of the computer interface, each child draws the numeric top-view coding from the picture of his or her own structure that the research team has formatted into a task card. Regardless of whether the children remember which Soma figures were used they know that the figures were assembled from two different ones since these pictures are their own creations. These semiotic codings become puzzles for their peers to decode using only the set of seven Soma figures. Examples are shown in Table 1. Some students drew 4-by-4 grids (as in C and D), which is a throwback to the grid provided by the Geocadabra Construction Box interface. Others (as in A, B and E) understood that rows or columns containing zeros were not needed. The child who created grid B enumerated the cubes in her structure, reflecting on work that had been done some weeks prior to this activity, to verify the volume of her structure.

To begin some puzzle solvers use enumeration strategies. A sum of 7 must mean that Soma #1, the only figure with 3 cubes, is used in combination with a 4-cube Soma figure, as in puzzles A and D. Some children are able to directly state which two Soma figures can be used to solve the puzzle. Others mentally visualize at least one Soma figure in the coding. The children quickly realize that selecting two random Soma figures is a time-consuming and ineffective strategy. A more effective scaffolding strategy discovered by some children is to use Soma #1 together with a loose cube. For example, in Puzzle B, the left side appears to be Soma #7 (see Figure 6). Then, the child places Soma #1 on the right side and holds a loose cube to its left as shown in Figure 6, to create Soma #6. Some children use one or two loose cubes in their interim solutions before they arrive at the assembly solution.

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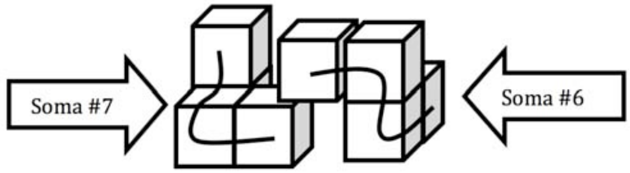

Figure 6. Using Soma #1 to solve puzzle B

While the 2D task cards were created with a 3D stimulus via the Geocadabra Construction Box computer interface, the reverse process, using the semiotic puzzle as a stimulus and the 3D model as a product, is performed without the aid of the computer interface. The capacity to visualize develops further when children are able to find more than one Soma figure combination to solve these particular problems as shown in puzzles A, C-E (Table 1). Two different children solved puzzle E using three different Soma figure combinations (solutions E1 and E2).

These examples are evidence of problem situations created and solved by the children based on the objects that were previously used as stimuli. Puzzle F is particularly interesting because this child created a unique extension of the original task using his two solutions, shown as E2. He realized that the two Soma figures were the same as the ones he had used for his own assembly shown in F to the far right. He stated that with "two movements" of the Soma figures he needed for puzzle E he could re-create his own assembly.

Following the puzzle posing activity, the teacher researcher displayed for the first time the formal SOC framework as shown in Figure 1 in order to elicit metacognitive interpretation of this graphic organizer from the children. She asked if anyone recognized any of the project's activities within the figure. One child pointed out that the puzzle creation activity (Table 1) started with the 3D model to create 2D pictures on the computer. Another recognized that the numeric top-view codings that they created from their 2D pictures belonged to the semiotic or abstract representation. She also stated that the class had moved from the semiotic to the 3D figures directly without the computer. When asked about the verbal description, one child said that they do it all the time when they think and share solutions together. The participating children have made sense of the SOC framework through the myriad of hands-on experiences over the past 5-6 months without prompting from more-knowledgeable adults. This is strong evidence that through carefully designed activities, using strategically chosen manipulatives, deep mathematical knowledge (Connell, 2001) and generalized abstraction (Presmeg, 1992) can develop in unprecedented ways. Furthermore, this occurrence demonstrates how the instructional team attends to and accommodates child-centered contributions to the development of the learning trajectory.

Conclusions

The capacity to move among the three SOC visual representations, namely, 3D models, 2D conventional pictures and semiotic representations, was developed through the Geocadabra computer interface. However, through extended problems created by the learners themselves, it is remarkable that they are able to move from the semiotic representation to the 3D model independently of the computer (Connell, M., personal communication, September, 2010). The research team considers this to be evidence of the children's growing ability to visualize as they move between semiotic and 3D assembly models. Through their different configurations the Soma figures provide a high degree of complexity to the instructional tasks and force the children to engage in mental transformations in ways that would not be possible if they used loose cubes.

The actions on objects (Connell, 2001) occur concretely with the 3D models, virtually through the Geocadabra interface and ultimately as mental imaging through the powerful problem-solving approaches developed by the research team. These evolve through the reflective practices enacted by the team immediately following each lesson (Sack, & Vazquez, 2011). The team has a very strong child-centered philosophy, coupled with the belief that children learn best when engaged in problems of their own creation (see also Connell, 2001).

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