CRITICAL MOMENTS IN GENERALIZATION TASKS. BUILDING ALGEBRAIC RULES IN A DIGITAL SIGN SYSTEM

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We report findings of research that deals with secondary students' interaction with a digital microworld, eXpresser. eXpresser is designed to support students in constructing general figural patterns of square tiles and expressing the algebraic rules that underlie them. Our focus is on the sense production that emerges as students use eXpresser. In order to analyze student productions, we adopt a semiotic perspective based on a broad notion of mathematical sign systems. We identify in the analysis, critical moments that illustrate the emergence of algebraic expression in eXpresser's alternative sign system. These moments are constituent elements of the process of deriving a general rule and support the development of the notions of dependent and independent variable.

Introduction

The findings of various studies that deal with pattern recognition and expression have led to conceiving of mathematical generalization as a gateway to learning algebra for pre-algebraic students. This approach seeks not only for students to become competent in recognizing regularities in numerical or figurative sequences, but that they also can imbue the corresponding algebraic expressions with meaning and sense (Noss & Hoyles, 1996, Mason, 2008; Bednarz, 1996; Kieran, 2007; Lee, 1996). Many research experiments have used this generalization approach to algebra while working with digital learning environments, and these have been shown to help students along that path, such as studies undertaken using Logo (Hoyles & Sutherland, 1989) and using spreadsheets (Sutherland & Rojano, 1993). Yet one of the greatest obstacles reported concerns the move from recognizing and analyzing a pattern to expressing it symbolically. The MiGen project (1) has developed a microworld, *eXpresser*, designed specifically to support students in generalizing rules based on structure, in this case of the structure of figural patterns of square tiles (Noss et al., 2009). The *eXpresser* microworld includes construction objects and a set of actions that can be carried out on them so that students can build models made up of these patterns and test their generality through animation.

In this paper we focus on the potential of students building a rule that represents the structural regularities of a pattern, while also being able to exploit a symbolism for exploring the performance of the pattern in general. This symbolism is closely tied to the nature of the sign systems made available by *eXpresser* for rule-building. The affordances of *eXpresser* features were directly used in the study reported here in order to design generalization and symbolization activities, which were investigated with groups of secondary school students, aged 11-13, in England and Mexico (2). The study aims to analyze the sense production processes that were triggered when students were on the path toward building a rule, which represented the structure

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of the patterns that together made their model. Episodes of in-class experimental sessions in which the students were interacting with *eXpresser* were analyzed from the basis of a semiotic perspective in which the notion of Mathematical Sign System (MSS) plays a central role (Puig, 2008).

Theoretical Perspective: Mathematical Sign Systems and Sense Production

Since mathematical texts are produced with a heterogeneous set of signs, the notion of Mathematical Sign System (MSS) dealt with in Filloy et al. (2010) and used in our analysis entails not only the signs considered as specifically mathematical, but also those pertaining to natural language, figures, diagrams, as well as the signs of digital learning environments. Hence that notion of MSS does not refer to a set of mathematical signs organized in a system, but rather to a system of signs –specific to mathematics or not- that is of a mathematical nature (i.e. the system is conceived as of a mathematical nature rather than the individual signs). It is crucial to take the system as a whole because the use (or uses) of the system is the responsible for meaning and sense making. Moreover the notion is not limited to socially established sign systems, but includes signs, sign systems or strata of sign systems that students produce in order to make sense of what is presented to them in a teaching model, in an interview situation or in a mathematics task.

Thus, sense production in mathematics activity comes about by way of chains of reading and transformations of actions that are undertaken on texts expressed in an MSS (Puig, 2003). Thus in the particular case of work with *eXpresser*, our interest lies in analyzing the sense production of students related to the expression of rules that correspond to figurative models built in an environment in which they interact with its MSSs through reading and transformation actions. In *eXpresser,* students have access to increasingly abstract set of MSSs, as evident below.

eXpresser: A Digital Microworld to Express Generality

In *eXpresser*, students are asked to construct a general model and to develop a rule that determines the total number of tiles in the model. The students construct the model by visualizing its structure and determining appropriate patterns of repeated building blocks that together make up the model. The students have to make explicit rules to calculate the number of tiles in each pattern, which are colored if and only if the rule is correct and then combine the rules for the different patterns to obtain the total number of tiles in the model. Of course there is more than one way to construct a model, each of which leads to a "different" rule for the same model.

Figure 1 shows a snapshot of the *eXpresser* with what is known as the Train-Track model already constructed (shown top left of the screen labelers My World). A C-shaped building block (A) has been created (by dragging and dropping separate tiles and grouping them) and is repeated by placing each repetition two squares across and zero places down (When making a pattern, the translations across and down for each repetition of a building block, as well as an initial number of repetitions has to be specified). When the C-shaped building block (A) is made, its properties are shown in an expression (B). The building block is repeated as many times as the value of a variable called 'Model Number' (D), in this case 4.

Patterns are colored by calculating and then allocating the *exact* number of colored tiles to its construction. In the case of the pattern made of C-shapes, using the expression for construction (B) and the number of times the building block is repeated, the rule for the total number of tiles in this pattern is '7×Model Number', in Figure 1 (E). As students build their constructions in 'My World', a second window is seen alongside (the 'General World'). This mirrors exactly My World until the student unlocks a number in their model, the metaphor for making a variable, at which point the General World chooses randomly a 'general' value for the variable. In Figure 1, the value of the Model Number in the General World is randomly set to 9, resulting in a different instance of the model (F). The General World is only colored when students express *correct* general rules in the 'Model Rule' area of the screen (G). Students cannot interact directly with the General World. They are, however, encouraged to click the play button (H) to animate their general model to test its generality.

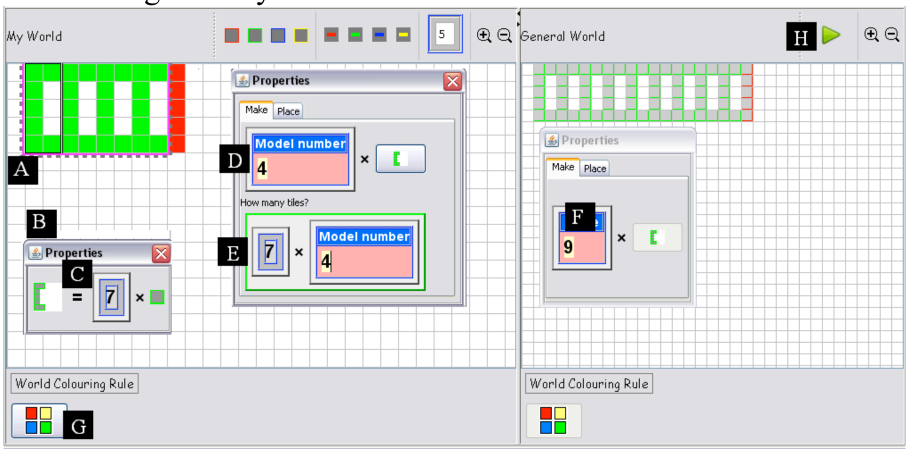


Figure 1. Constructing a model and expressing its rules in *eXpresser*. (A) Building block. (B) Expression of construction of the building block. (C) The number of tiles of the building block. (D) The number of repetitions of the building block. (E) Number of tiles required for the pattern, with general rule. (F) Any variable used in 'My World' takes a random value in the 'General World'. (G) A general rule that expresses the total number of tiles in the whole model. (H) Play button for animating patterns.

The construction objects in *eXpresser* are structurally related to the elements of the symbolic rule as follows: a) the number of building blocks is associated with the variables and constants; b) the number of tiles in each block is associated with the coefficients of the variables in the expression; c) the patterns constitute the terms of the expression; d) and the model is associated with the complete expression. In particular terms in the task referred to in this article, the students were asked to build a model associated with a symbolic expression that had linear and constant terms. In other words, the model had to be made up of several patterns that together would serve to represent a Train-Track model, which, if properly built, could be "extended" as student wanted. The students were asked to use different colors for each of the patterns that made up the model, in order to highlight the way they perceived the structure. The students were also required to answer questions that are aimed at provoking reflection on the model-formula structural relations that they had constructed.

Experimental Work with eXpresser

The data presented in this paper refer to the work of 11 to 13 year old students (14 Mexican and 22 English students) who began to study algebra in secondary school. Although the students worked on several activities throughout the study, this report only includes an analysis of the Train-Track activity. Data was compiled from videotapes of classroom sessions, written reports of the students on worksheets and computer files produced by the students registering their interactions with the software. Our attention was focused on analyzing the role of the different sign systems proposed in *eXpresser* in the sense production processes. In England, the task was presented fully computerized and students answered some reflective questions presented through digital worksheets. In Mexico, the students worked with *eXpresser* and with paper and pencil worksheets.

Results – MSSs in eXpresser

In building models, the students moved from working with MSSs associated with visual perception to more abstract MSSs, such as numerical and in some cases algebraic MSSs. During the process it was possible to see how the students made sense of the elements of a symbolic expression by associating them with the specific elements that made up their figurative model. Thus the generalization processes consisted of searching for a structure that could be transformed from the figurative model to a symbolic expression by means of the structure of the task-pattern, and then a rule could be derived based on the perceived structure. Data analysis made it possible to identify critical moments of generalization activity, which characterized aspects of work in *eXpresser* related to the nature of the different MSSs available in the setting. These aspects include, *inter alia*: a) constructing building blocks from unit tiles; b) unlocking numbers in order to vary them; c) numerical articulations among the different patterns that make up the model; d) building model rules using the *eXpresser* language; e) building different patterns based on the perceived structure to produce different models; f) analyzing the relations between the particular construction and the general models that it is possible to see in *eXpresser*. In this paper we focus only in the affordances of *eXpresser* that were designed to aid the transition from pattern construction to arithmetic expression and its transformation to algebraic expression (aspects b) and d), for which we have used analyses of student working on the Train-Track activity.

Building Blocks, Patterns and Models

A building block is an object that belongs to a MSS that is more abstract than that of individual tiles, albeit both are of a figurative nature. First, the students must become aware of the need to determine a building block in order to make a pattern. Analysis of the videotapes and of the files saved automatically by the software when the user interacts with it enabled us to follow up on student actions in their attempts to build their patterns and models. Although the students had no trouble defining the building blocks, records of their work in both countries showed that some had a tendency to simply produce the pattern tile by tile. In this case, they were helped by redirect their attention to the building block they had been in fact implicitly repeating while dragging tiles, thus helping the students identify their own building blocks (see Noss et. al., 2009). Ten different forms of constructing the Train-Track model were identified, see Table 1.

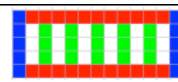
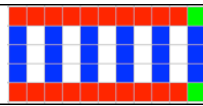
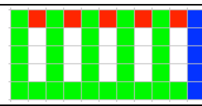
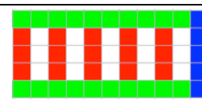
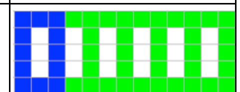
A	B Locked numbers	C	D	E
1Mexican & 2 English students	4 Mexican & 3 English students	3Mexican Unlocked numbers students	1Mexican student	1Mexican & 9 English students
				
Two patterns: $5n + 2(n - 1)$	Three patterns: $3n + (2n – 1) + (2n – 1)$	Three patterns: $5n + (n – 1) + (n – 1)$	Five patterns: $3n+ (2n+1) + (2n+1) + 5 + 5$	Two patterns: $7n + 5$
F	G	H	I	J
1 English student	1 English student	1 English student	4 English students	1 English student
				
Three patterns: $5n + 2n + 5$	Three patterns: $4n + 3(n+1) + 2$	Three patterns: $6n + n + 5$	Three patterns: $4n + 3n + 5$	Two patterns: $7n + 12$

Table 1. Different forms of constructing the Train-Track by students participating in the study. To better communicate each model's structure, in the bottom line the corresponding algebraic rules are shown (*n* is the number of one of the building blocks in the model). It should be noted that this not mean that the participant students could produce such an expression in the algebra MSS.

From Arithmetic Expressions to Algebraic Rules

One important step in the generalization process toward the use of an abstract MSS is that of deriving a symbolic rule (arithmetic or algebraic) that makes it possible to obtain the overall number of tiles in a pattern (dependent variable) in terms of the number of building block repetitions (independent variable). The microworld provides immediate visual feedback when, for a given pattern, the production rule is correct: that is by coloring all of the tiles used in the particular construction (see Figure 2).

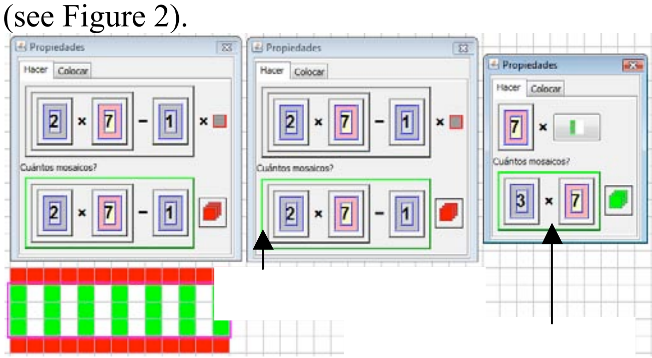


Figure 2. Arithmetic rules for the three patterns that constitute the model (top (red) line), bottom (red) line and vertical (green) line)

Different types of numerical expressions (texts) can be seen in the image on the left of Figure 2, including: locked numbers (constants) and unlocked numbers (generalized numbers) in which the values can change. Identification of the role played by these types of numbers in the figurative model is essential to the construction of algebraic rules. Although indicating the overall number of tiles that constitutes a pattern is sufficient for it to be colored, a rule must be provided in order for the pattern to remain colored for *any* number of repetitions of the building block (so that the Train-Track can, in turn, be extended or reduced). Most students in both Mexico and England used initially a specific number to indicate the number of tiles in each pattern. Despite introductory tasks in *eXpresser*, the persistence of this student strategy shows (unsurprisingly given previous research) a common reluctance to articulate a relationship and to express it in a general way. To overcome, the teacher (3) had to intervene and encourage students to use a building block. The intervention relies on repeatedly changing the variable in the number of repetitions of the building block and therefore 'messing-up' the coloring of the pattern– drawing on the vocabulary used by students in a dynamic geometry setting. The visual feedback challenged students to find a general rule that according to Gregg (one of the students in the English trials) "tells it [the *eXpresser*] how many tiles it needs for any number in the pink [unlocked] box". However, some students (9 in Mexico) when answering the worksheets sometimes but not always made a numerical formula for some of the patterns in the model. For example, Figure 3 shows the work of Sofia indicating two formulas: one with an operation (7×5) and other with a constant number (6) in the case of model C in Table 1. Sofia also forgot to indicate the arithmetic expression for one of the patterns, since the last one was not built with the pattern construction tool, she used the tool copy-paste to include it in the model.

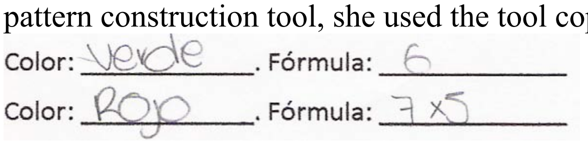


Figure 3. Construction of one numerical formula and one constant to indicate the number of tiles in two patterns (Verde – Green; Rojo – Red)

Eventually, most of the students managed to make symbolical formulae for their patterns, though many were isolated and not interrelated. For example, in the worksheet in Mexico, Diego constructed a formula with two variables (a×5 + 2b), for the model (C) in Table 1 he did not seem to see the relationship between a and b ($b = a - 1$) and it was only after teacher intervention that he (and another two students) were able to construct a formula with one independent variable (see Figure 4).

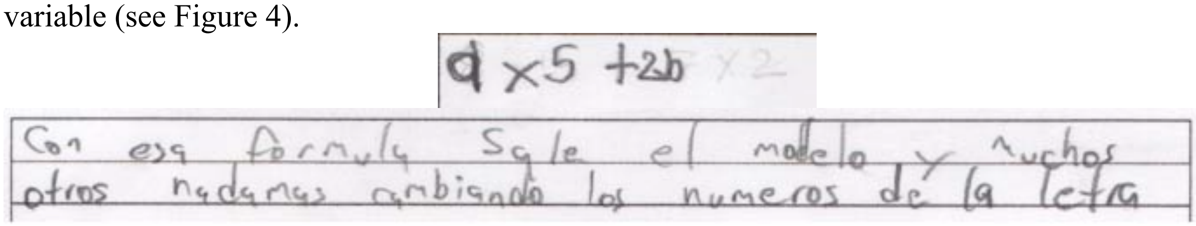


Figure 4. A student's work on expressing the formula to calculate the number of tiles in the Train-Track model. Student writing: "With this formula I get the model and many more when I change the numbers of the letter"

An effective intervention capitalizes on drawing students' attention to the General World. Models that use patterns with more than one independent variable, are 'messed-up' in the General World since, for each variable, the microworld chooses random values likely to be different so the models do not maintain their structural coherence. Figure 5 shows Connor's work, an English student; he has unlocked both the 5 and the 3 as the number of repetitions in his two patterns, but has not specified any explicit relationship between them. The teacher asked him many times to choose several different random values for the number of green building blocks and find the number of red building blocks. Connor painstakingly repaired the model each time when it was no longer colored. He had implicitly identified a relationship but could not express it in general terms. After several repetitions of the same prompt, when the value of repetitions of the green block was 8 and that of the red had to become 15 (2×8 – 1), Connor verbalized his rule: "You times the 8, you times the number of green tiles by 2 […] and then take away 1", and used *eXpresser* to represent explicitly the relationship between the two patterns in his model.

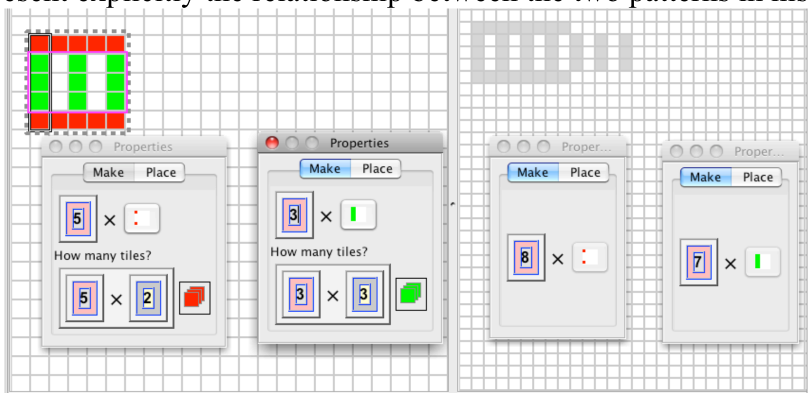


Figure 5. Connor's model is messed-up in the General World

In *eXpresser*, numbers are perceived as objects to which a value can be assigned, that can be locked or unlocked and given a name. Hence, in their capacity of being concrete objects, numbers are associated with one single screen position. If that number is required in order to construct a rule (simply being represented by the same variable is sufficient in the MSS) then it must be a part of two different relations, thus forcing the student to make a copy of it and not simply defining another number (or variable) that has the same value. As such, in Figure 2, the object-number shown with a numerical value of 7 was copied five times in order to construct and relate the rules for the three patterns that make up the model. Determining this fact is one of the most important characteristics of building a formula in the MSS of algebra because the subsequent symbolization only consists of assigning a name to that number.

Differentiating specific numbers from general numbers and number-objects means that *eXpresser* substantially changes the activity of mathematical expression of patterns and thus makes the passage from generalized numbers to use of letters more natural within the digital environment. As such, students are able to reflect upon establishing relations among objects that, despite belonging to different strata of MSSs, all have a figurative referent that enables the students to imbue those relations with sense. We assert that this characteristic made it possible for the majority of students participating in the study to express a rule corresponding to the Train-Track model in an algebra like MSS. Figure 4 shows part of the worksheet of Diego on which he has expressed the formula for calculating the overall number of tiles in the Train-Track model, as well as his interpretation of that formula (expressed in words).

Students in England were asked to reflect on their derived rules in *eXpresser* by writing some arguments to support their rules' correctness or not. A 12 year old student, Nancy, whose model/rule is shown in Figure 6, wrote: "My rule is correct because each "block" has 7 squares. So however many blocks there are, there are 7 squares for each one so you multiply the number of blocks by 7. But, at the end there is another block to finish the pattern off. In this block there are 5 squares so you add the number of squares (the blocks multiplied by 7) to the final block (the 5 squares). This rule should apply to this pattern each time". Nancy showed a good understanding of each term in her derived rule and seemed to understand the generality of her

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rule. She was able to identify how the coefficients in her rule were derived and expressed clearly what her variable was, i.e. the number of blocks. In her own words, the term ‘any’ was referred to as ‘however many’.

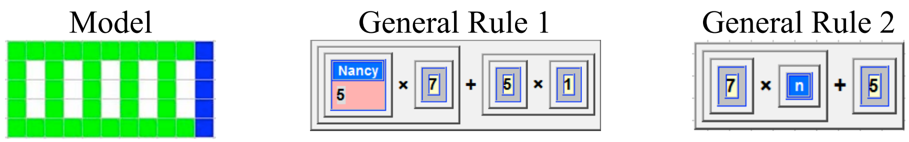

Figure 6. Nancy’s model and rules in eXpresser

Another 12 year old student, Connor (model (B) in Table 1), wrote: “My rule is correct because it worked without fault and the color stayed on the General World. (6×3) + [(6×2)–1] ×2 this is my rule. The 6×3 is the green tiles and how many of the green blocks we need times the how many tiles are in each block. The 6×2–1 is the red tiles and the -1 because otherwise the red tiles would go over how many you need and ×2 at the end is because I’ve got 2 lots of the red”. Even though Connor might think generally, he still used the specific value 6 for his variable when he wrote down his general rule- a generic example maybe. It is evident that he also had a clear understanding of each term in his rule supporting his rule’s correctness by reference to the feedback from the *eXpresser*, i.e. his model stayed colored for any model number.

Discussion

The availability in the environment of an increasingly abstract set of MSS strata to work with specific numbers, general numbers and number-objects made it possible for the students participating in the study to formulate rules that generated different patterns and to inter-relate them by way of one single independent variable. This was we argue a critical moment in a generalization task and helped students to develop and distinguish the notions of independent and dependent variable. We assert that the relationship among these generative rules is the result of the acts of reading and transforming that the students undertake on their own productions in *eXpresser* when trying to formulate one single rule to calculate the total number of tiles in the figurative model that was made up of different patterns. Production of sense arises in those acts with respect to expression of the rule in an algebra-like MSS. Nonetheless some of the students were able express generality in their own and *eXpresser*’s language, but still found it difficult to express it algebraically, a move supported by later collaborative tasks.

Endnotes

1. The MiGen project is funded by the ESRC/EPSRC Technology Enhanced Learning programme; Award no: RES-139-25-0381). For more details about the project see http://www.migen.org.
2. The experimental work in Mexico is funded by CONACYT (Grant No. 80359).
3. In England, the students were supported by either the teacher or the intelligent support component of the MiGen system, which provides prompts based on students’ interactions and helps them reflect and decide about their next step in solving the task

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