MIDDLE SCHOOL ALGEBRA FROM A FUNCTIONAL PERSPECTIVE: A CONCEPTUAL ANALYSIS OF QUADRATIC FUNCTIONS

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This study presents a model of three conceptual advances in understanding quadratic functions based on a teaching experiment with 6 8th-grade students. Using a covariation approach, students investigated quadratic growth as a coordinated change of x- and y-values. Qualitative analysis yielded three major shifts in students' understanding: a) developing an understanding of the first and second differences for y as rates of growth with Δx implicit; b) explicitly coordinating Δx with the constant second differences; and c) coordinating covariation and correspondence views through meanings for the parameter "a" in $y = ax^2$.

Objectives

Calls for "algebra for all" have grown in frequency in recent years, influencing district policies nationwide. For instance, the National Mathematics Advisory Panel (2008) recommends that districts prepare to enroll increasing numbers of students in algebra by Grade 8. Similarly, the NCTM (2000) Principles and Standards calls for including algebraic ideas throughout the K-12 curriculum and recommends that middle school students in particular focus on learning concepts in algebra. Successful implementation of "algebra for all", however, depends on finding ways to help students understand fundamental algebraic concepts such as equality, the use of variables, and functions. Research suggests that traditional courses focused on strategies for manipulating symbols, simplifying expressions, and solving equations yield poor results in overcoming students' well-documented difficulties in understanding algebraic relations (e.g., Knuth et al., 2006). These limitations have led to efforts to expand notions of what constitutes school algebra, and one major set of recommendations emphasizes a functional perspective as a central concept for organizing algebra instruction (Schliemann, Carraher, & Brizuela, 2007), with the early introduction of functional relationships in the elementary and middle grades. Placing functions at the center of algebraic reasoning can support students' abilities to make sense of quantitative situations relationally and provide an important foundation for future success in mathematics.

Given the importance of functional understanding for developing algebraic reasoning, one of the critical challenges remains better understanding how students' early function conceptions develop. As Asquith et al. (2007) noted, the challenges in learning more about students' reasoning "are particularly relevant at the middle school level, at which time the transition from arithmetic to algebraic thinking is arguably most salient" (p. 250). This paper presents an investigation of middle school students' emerging understanding of quadratic growth, presenting a model of three conceptual advances students experienced when studying quadratic function. It closes with a discussion of the implications for algebra understanding.

Students' Understanding of Quadratic Functions

Quadratic functions represent the basis of the more advanced mathematics to come at the secondary level and as such can act as a transitional topic for supporting students' developing algebraic reasoning. However, attempts to effectively introduce quadratic relationships have proved difficult. Students struggle to understand the role that the parameters "a", "b", and "c" play in $y = ax^2 + bx + c$, and have difficulty describing the effects that changing the parameters

can have on a function's graph (Zazkis, Liljedahl, & Gadowsky, 2003). Studies have also documented students' tendencies to inappropriately generalize from linearity in order to make sense of quadratic relationships (Chazan, 2006), including using linear interpolation and attempts to find a slope and a point (Ellis & Grinstead, 2008; Eraslan, 2007). The phenomenon of generalizing from linearity suggests a need to consider introducing non-linear functions earlier in students' algebraic reasoning. One aim of this study was to identify the concepts that middle school students developed when investigating quadratic relationships from the perspective of reasoning about quantities.

Conceptual Analysis and the Development of Models of Students' Thinking

Conceptual analysis as a tool in mathematics education can be employed to satisfy a number of different goals. One can develop a conceptual analysis in order to specify the mental operations required to obtain a particular set of concepts or to analyze ways of understanding a body of ideas based on describing the coherence between their meanings (Glasersfeld, 1995; Moore, 2010). Thompson (2008) identified four uses of conceptual analysis: a) to build models of what students actually know at a specific time and in specific situations, b) to describe propitious ways of knowing for students' mathematical learning, c) to describe ways of knowing that might be problematic to students' understanding of important ideas, and d) to analyze the coherence of various ways of understanding a body of ideas. The purpose of this study is compatible with (a); its aim is to build a model of what students understand about a particular type of quadratic growth. In so doing, the model introduces advances in understanding quadratic growth that may be favorable for fostering a deeper understanding of functional relationships.

Following Glasersfeld's theory of radical constructivism (1995), the analysis presented here is based on the understanding that a student's knowledge is fundamentally unknowable, and thus any conceptual model is simply a researcher's tool for making sense of the student's mathematics. From this perspective it becomes important to refine tentative models over time. The use of the teaching-experiment methodology (Steffe & Thompson, 2000) supported the creation, testing, and revision of models of students' mathematics over multiple iterations. One aim of the teaching experiment was to investigate the viability of an introduction to quadratic function that emphasized the covariation approach (Confrey & Smith, 1995) within a quantitatively-rich setting. Within this approach, students examine quadratic growth as a coordinated change of *x*- and *y*-values. An open question was whether students could make quantitative sense of the phenomenon of constant second increases for *y* coordinated with uniform increases for *x*, and whether this understanding could support a more robust view of quadratic function that could be meaningfully connected to the correspondence rule $y = ax^2$.

Methods

Participants and the Teaching Experiment

The study occurred at a public middle school with 6 8th-grade students, whose teachers identified them as high performers (2 students), medium performers (2 students), or low performers (2 students). Students across a range of performance were included in order to create a heterogeneous group in terms of mathematical backgrounds, knowledge, and skills. The students participated in a 15-day teaching experiment, which met for 1 hour a day. The students worked with a computer simulation of growing rectangles in Geometer's Sketchpad, in which they could manipulate the size of the rectangle. By adjusting the rectangle's height, the length would adjust automatically, preserving the height/length ratio (Figure 1):
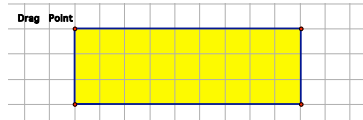
Figure 1: Image Capture of a GSP growing rectangle script

The students explored relationships between the heights, lengths, and areas of various rectangles by using scripts in Geometer's Sketchpad and by creating their own drawings, tables, graphs, and equations. The author taught the sessions, which were observed by two project team members who offered reactions and commentary after each session.

Data Sources and Analysis

All sessions were videotaped and transcribed, and additional data sources included students' written work and the author's written reflections on each session. Data were analyzed via the constant comparison method with an open and axial coding technique. Retrospective analysis (Steffe & Thompson, 2000) of the videotapes supported the creation of an initial model of each student's evolving understanding of quadratic growth. The development of the initial models of conceptual change led to the identification of 4 major categories of interest across all students: (a) students' meaning for the first differences; (b) students' meaning for the constant second differences; (c) the coordination of the increases in height with the increases in area; and (d) the relationships between the constant second differences, the rectangle's dimensions, and the parameter "a" in $y = ax^2$. Identifying the students' operations for each category led to the development of the model presented below.

Results: A Three-Stage Model of Students' Conceptual Development

Three major conceptual advances occurred during the teaching experiment that shifted the students' evolving understanding of quadratic growth. Figure 2 provides an overview of each of the conceptual advances and the sub-categories of shifts that occurred within each stage.

Stage 1: Understanding differences as rates of growth with Δx implicit	*1a*: 1st Differences as the rate of growth of the area
	1b: 2nd Differences as the rate of rate of growth of the area
Stage 2: Understanding differences as rates of growth with Δx explicit	*2a*: There is a relationship between Δx and the 2nd differences
	2b: Connecting the 2nd differences to the rectangle
	2c: Identifying how Δx is related to the 2nd differences
Stage 3: Coordinating the covariation and correspondence views	*3a*: Viewing "a" as change in length per 1-unit change in height
	3b: Viewing "a" as the ratio of the change in length to the change in height
	3c: Understanding "a" as ½ the 2nd differences (Δx implicit)
	3d: Understanding "a" as ½ the 2nd differences (Δx explicit)

Figure 2: Three stages and sub-categories of conceptual advances

The following table is one student's record of a 2 cm by 3 cm rectangle:

Figure 3: Table of length/width/area values of a growing rectangle

All of the students' discussions about quadratic growth relied on the growing rectangles context. The students created tables of data to record the height, length, and area values they observed in Geometer's Sketchpad such as the one seen in Figure 3. The students' propensity to represent data in well-ordered tables in which the height or the length values increased by uniform 1-cm increments encouraged a focus on what is often referred to in textbooks as the "first differences" and the "second differences" for *y*. The first transition the students experienced was one in which they shifted from understanding these values as differences to understanding them as rates of growth.

Stage 1: Understanding Differences as Rates of Growth With Δx Implicit

1ˢᵗ differences. The students initially focused only on the differences between successive area values without coordinating with the way the height or the length grew. For instance, when describing the first differences for height/area table of a square that grew by 1-cm increments, Jim explained, "it goes 1 and then 3 and then 5 and then…and then you go to 3, 5, 7,…just keeps going." Attention to coordinating the growth in area with how the height grew was absent from Jim's description. The teacher-researcher asked the students to draw diagrams of growing rectangles that depicted the increases in area. Figure 4 shows Ally's 2 x 3 rectangle that grew to become a 3 x 4.5 rectangle and then a 4 x 6 rectangle:
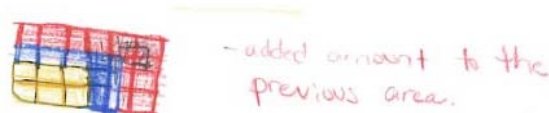

Figure 4: Ally's depiction of the additional area produced as the rectangle grows

Ally explained, "So we added 7.5 to this part [pointing to the 2 x 3 rectangle], we added 10.5 to this part [the 3 x 4.5 rectangle], and then, because it's the difference, it's [the second differences] how many more squares you had to add to this one [the 3 x 4.5 rectangle], instead of compared to this one [the 2 x 3 rectangle]." In this depiction Ally began to coordinate the number of squares making up the additional area to each time the rectangle "grew". The increments were not explicit for Ally, but she and the other students began to attend to the fact that the first differences represented the growth of the area for each increase in the rectangle's size. The next day, Jim more generally stated that the first differences represented "How many new squares it's gaining every time it grows." Jim's use of the term "every time" suggests that he was

coordinating the growth in area with some growth in height and length, but did not explicitly attend to the unit of growth.

2nd *differences*. The students initially described the second differences as "the second level sort of thing", or "the new outside area." Ally's picture in Figure 4 helped her explain that "every time it grows it adds 3", but it was unclear whether she saw this as the rate at which the rate of growth of the area grew.

A continued emphasis on creating drawings helped the students clarify the meaning of the second differences. For instance, Bianca explained that they represented "the area of the next shape minus the area of the previous shape", or the "amount added to the amount added to the area." Jim noted that the students should name this value, stating, "I think we should give it names, like, the amount added to the amount added is so confusing!" Ally suggested "Difference in the Rate of Growth" and the students settled on this term, eventually shortening it to the "DiRoG". Jim characterized the DiRoG as, "So the rate of rate of growth is how many square units it's gaining from the rate of growth."

When the students created a table for the height, length, and area of a 1 x 2 growing rectangle, Jim found the DiRoG to be 4 square units, and then exclaimed, "it's going up by the rate of the rate…the rate that the rate of growth is growing!" Elaborating, he said, "When I add that new shelf thing [referring to increasing the size of the rectangle], there's 4 left over instead of 3." At this point Jim began to connect the DiRoG to an increase in the rectangle's size, but the unit of increase remained implicit. The connection to the manner in which the rectangle grew was limited to conceiving of how the area grew "each time", rather than for a specific value for which the height or length increased.

Stage 2: Understanding Differences as Rates of Growth With Δx Explicit

Δx and the 2nd differences. If the students were not coordinating the DiRoG with the value of Δx, they would likely not anticipate that the DiRoG would change for a table in which Δx was something other than 1. In order to test this prediction, the teacher-researcher asked the students to create tables for a 2 cm x 5 cm growing rectangle, anticipating that some students would increase the height value increased by 1 cm, and others would increase the height value by 2 cm. This did occur, for instance, Daeshim created a table in which Δx was 2 cm, and Jim created a table with Δx as 1 cm.

The students argued about whether the DiRoG should be 5 cm^2 or 20 cm^2 until Jim realized that it depended on Δx: "He's [Daeshim] going by 2's, but I'm going by 1's." After a class discussion in which the students agreed that the DiRoG could legitimately be 5 or 20, Jim asked, "So your rate of growth can change no matter what?" At this stage the students understood that the DiRoG depended on how the rectangle grew, but had not yet determined how the DiRoG was dependent on Δx.

Connecting the 2nd differences to the rectangle. Because the students struggled to determine how Δx would predict the DiRoG, they decided to consider the DiRoG in relationship to the rectangle's dimensions. Daeshim drew a picture to show that the DiRoG would be twice the area of the original rectangle for a 4 cm by 14 cm rectangle (Figure 5):
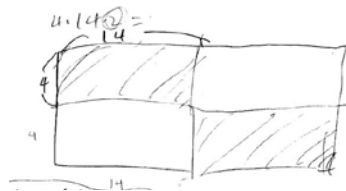
Figure 5: Daeshim's drawing of a growing 4 cm x 14 cm rectangle

Daeshim iterated the height so that the next rectangle became an 8 cm by 28 cm rectangle. He explained that this effectively created an additional area equivalent to 3 of the original rectangles, and repeating this process again would produce an additional area equivalent to 5 of the original rectangles, so the DiRoG, representing the difference in the rate of growth of the area, would be equivalent to the area of two of the original rectangles. Daeshim produced a corresponding table in which the Δx value was 4, and the DiRoG was 112 cm^2.

In contrast, Jim decided that the DiRoG would be equivalent to twice the length of the rectangle when the height was 1 cm: in this case, that would be 7 cm^2. He explained, "You reduce the height until 1, and then you can just multiply the area, or length, by 2. Because at 1, length and area are the same." He was aware that this depended on the rectangle growing in 1 cm increments for the height. Bianca agreed with Jim's method, but characterized it differently, stating "It's the length when the height is 2." Jim and Bianca imagined a rectangle that was a 1 cm x L cm rectangle. In this case reducing a general H x L rectangle to 1 cm for the height would produce a length of L/H. The area of that rectangle would be L/H, and thus the DiRoG would be $2 \times$ L/H, which is equivalent to twice the length (or area) of a rectangle when its height is 1 cm.

Δx and the 2nd differences. The different table configurations for the 4 cm x 14 cm rectangle led to the two DiRoG values, 7 cm^2 and 112 cm^2. Tai explained that both values were correct, "Because one the height is growing by 4, and for the other one, height's growing by 1." Daeshim then explained that when the DiRoG, 7, is multiplied by the square of the Δx value in the other table, 4, the result is 112: $7 \times 4^2 = 112$. In general terms, if the Δx value increases by h units instead of 1 unit, the DiRoG must be multiplied by h^2. After working with many different table configurations, the students decided that this relationship was true, but were not able to explain why. It was at this point that shifting to a correspondence view became necessary to move the students' thinking forward.

Stage 3: Coordinating the covariation and correspondence views

Viewing "a" as the change in length per 1-unit change in height. The students examined a table of height and area values and had to predict the area when $h = 82$ (Figure 6).

Figure 6: Student's work on a far prediction problem

The students introduced a third length column and found length values by dividing the area by the height. Tai explained, "The area divided by the height is the length. And if you can find out the length, then for this then you can find out the area." The constant increase of 4.5 cm in the length helped the students create a general strategy. Jim explained, "It would go over 4.5 for every time you go up the height 1." Jim determined that he could find the area by multiplying the

length by the height, i.e., "$n \times 4.5 \times n$", which the students then shortened to $4.5n^2$. Jim and the other students' ongoing attention to coordinated changes even as they identified a general correspondence rule marked the beginning of their coordination of the covariation view and the correspondence view.

Viewing "a" as the ratio of the change in length to the change in height. Once the students created equations in the form $y = ax^2$, they began to examine connections between "a" and the quantities height and length. The parameter "a" can be thought of as the ratio of the rectangle's length to its height. However, this view did not gain purchase with the students, who were entrenched in a dynamic view, preferring to think about how the heights, lengths, and areas changed as the rectangle grew. Tai explained, "The number in the front is always the difference in the length divided by the difference in the height." Daeshim formalized this as "dL/dH", where "dL" and "dH" referred to the constant differences in successive length and height values. The students' ability to relate "a" to a coordinated change in height and length values served to further connect the covariation and correspondence views.

Understanding "a" as ½ the 2nd differences (Δx implicit). The students noticed that the "a" in $y = ax^2$ was half the DiRoG. This is true only for tables in which Δx is 1, but the students did not initially attend to this limitation. After creating the correspondence rule $A = 0.75h^2$, Jim explained, "The difference of the rate of growth, half of that is here [points to .75]." After working with multiple tables, the students agreed with Jim's conclusion and Bianca formalized it as "DiRoG/2 = a". At this point, the students understood the parameter "a" in two ways: as half the DiRoG (although this depends on the height growing in 1-cm increments), and as the ratio of the change in length to the change in height. This led Bianca to realize that they could determine the DiRoG by finding $\Delta L/\Delta H$ and multiplying it by 2. However, the students could not easily explain why their generalizations were true, and they were not aware of the fact that their generalizations were limited to the case in which height increased by 1 cm.

Understanding "a" as ½ the 2nd differences (Δx explicit). In order to help the students re-focus their attention on the change in height values, the teacher-researcher introduced tables with varying Δx values. The students examined a new table with a Δx value of 4 in which the correspondence rule was $y = 4.5x^2$. The students were instructed to consider, "What does the 4.5 have to do with the DiRoG?"

It was through a re-consideration of the quantities height, length, and area that the students eventually began to connect the DiRoG, the "a" value, and the Δx value. The students initially predicted that "a" would be equal to DiRoG/2. They quickly determined that this was incorrect and at that point recalled Daeshim's conjecture that if the Δx value increases by h units instead of 1 unit, the DiRoG must be multiplied by h^2. Bianca realized, "Length over height! It's length over height, times DiH (Δx) squared, times 2 equals the DiRoG of the area." Bianca realized that "a" was the ratio of the length to the height of the rectangle. She knew that typically, twice this value is the DiRoG, but she also realized she needed to compensate for the Δx value being 4 instead of 1. Multiplying by Δx^2, which she called "DiH", provided the correct compensation. Eventually, the students formalized this connection as "a = (DiRoG/2) × DiH^2".

Discussion

The students experienced little difficulty transitioning to tables that did not increase by uniform height values. The images of the changing height and length values in the growing rectangle supported their ability to make sense of these tables and create correspondence rules. Moreover, the students were able to make meaningful connections between correspondence rules

and graphical representations by relying on images of coordinated changes between height, length, and area. For instance, all of the students correctly predicted that the parabola for $y = 5x^2$ would be narrower than the parabola for $y = 0.5x^2$, because the former represented a larger rectangle at each specific height value that would add more area with each increase in height.

The students' conceptions about quadratic growth were not without limitations. It was challenging for them to coordinate the growth in area with corresponding growth in height and length, and the students did not appear to reach a point at which they viewed the quantities as varying continuously rather than in repeated discrete increments. Nevertheless, the students developed a number of important threshold concepts (Meyer & Land, 2003) during the course of their investigations. They conceived of the first and second differences as rates of growth, and coordinated the growth of area with growth in height and length values. Their coordination of the covariation view with the correspondence view also fundamentally changed their understanding of the role of the parameter "a", and aided their abilities to create meaningful correspondence rules to represent data.

Lobato et al. (2009) remarked that a common view of conceptual analysis is that it should be based on analyzing the understanding of mathematically sophisticated adults, if participants are included at all. When student thinking is mentioned, its value is relegated to providing a window into the psychology of mathematics rather than as a source for articulating the substance of mathematics. But analyzing student reasoning, particularly at the middle-school level when students confront complex functional relationships for the first time, can inform the construction and refinement of a set of conceptual learning goals. The results of this study suggest that a covariational approach to quadratic relationships can provide a foundation for understanding the nature of quadratic growth and can support a meaningful transition to correspondence relationships.

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