A PROCESSES APPROACH TO MATHEMATICAL KNOWLEDGE FOR TEACHING: THE CASE OF A BEGINNING TEACHER

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This study examines the connection between mathematical knowledge (described as a teacher's engagement in mathematical processes and actions on the products of those processes) used by a beginning secondary mathematics teacher (Fiona) in her personal mathematical problem solving and the mathematics in which she engaged her students in her classroom instruction. This Process and Action approach involved analysis of Fiona's use of four mathematical process/product pairs (justifying/justification, generalizing/generalization, defining/definition, and representing/representation). Two themes arose in the analysis of interview and classroom observation data: (a) Although able to do so, Fiona did not regularly engage in processes in her personal mathematics or classroom mathematics, and (b) Fiona focused on selected features of a product or mathematical object rather than attending to all relevant features.

Keywords: Teacher Knowledge; Mathematical Knowledge for Teaching

Purposes or Objectives of the Study

Research interest has recently burgeoned regarding the relationship between teachers' mathematical knowledge and the ways that that knowledge impacts what happens in the classroom. Of particular interest is how teachers' knowledge affects both what teachers do and what students learn. Over the years, researchers (e.g., Eisenberg, 1977; Hill, Rowan, & Ball, 2005; Monk, 1994) have investigated the relationship between teacher knowledge and student achievement, typically using proxies for teachers' mathematical knowledge. These studies have often found that teacher knowledge is related to student achievement, but they have not shed light on how teacher knowledge affects what is happening in classrooms. This question has received much less attention from researchers, and many of the studies of that relationship do not separate content knowledge and pedagogical content knowledge (e.g., Hill, Ball, Blunk, Goffney, & Rowan, 2007; Lehrer & Franke, 1992; Swafford, Jones, & Thornton, 1997), leaving one to wonder about the effects of content knowledge itself on instructional practice. Studies that examined the relationship between content knowledge and classroom practice (e.g., Baumert et al., 2010; Rowland, Martyn, Barber, & Heal, 2000; Tchoshanov, 2011; Wilkins, 2008) have found relationships, but these studies focused on narrowly defined aspects of classroom practice (e.g., cognitive demand of tests and homework, students' opinions about instruction, teachers' self-reports of reform practices) and used written tests of predetermined categories of teacher knowledge (e.g., high and low content knowledge, cognitive type of content knowledge). Although the studies identified relationships, they did little to explain why these relationships might have occurred. The study reported here used extensive sets of interviews and observations focused on teachers' mathematical knowledge and its use in the classroom to characterize and explain the relationship between a teacher's mathematical knowledge and classroom practice. This study addressed the question of what characterizes a beginning secondary mathematics teacher's engagement in personal mathematics and classroom mathematics and the relationship between them.

Perspective

We describe mathematical knowledge in terms of four mathematical processes and their respective products: defining, justifying, generalizing, and representing, and actions on (or uses of) definitions, justifications, generalizations, and representations (Zbiek, Peters, & Conner, 2008). Using the processes and products to characterize a teacher's mathematical knowledge allows us to examine the mathematics demonstrated in his/her problem solving (*personal mathematics*), the mathematics in which the teacher engaged his or her students (*classroom mathematics*), and the relationship between the teacher's personal and classroom mathematics. A second affordance of using mathematical processes is that it transcends both mathematical content areas and grade levels, since the use of these processes and products is not dependent on either of these factors. Examining mathematical knowledge as knowledge evidenced by engagement in mathematical processes allowed us to examine mathematical knowledge over a period of three and a half years across three different content areas.

Methods

Our data consist of five task-based interviews and 16 teaching observation cycles of a secondary mathematics teacher, Fiona (a pseudonym). At the beginning of data collection, Fiona was enrolled in a secondary mathematics teacher certification program at a large Mid-Atlantic university. She was one of several in the program who volunteered to participate in the study. The task-based interviews (*Area*, *Count*, *Cube*, *Wrap*, and *Defining*) were conducted during Fiona's teacher preparation program for the purpose of understanding her use of mathematical processes and products in her personal mathematics. To understand Fiona's use of processes and products in her classroom teaching, teaching observation cycles were conducted during her student teaching (pre-calculus), first-year teaching (algebra), and second-year teaching (geometry). An observation cycle consists of a pre-observation interview, an observation, and a post-observation interview.

All the task-based interviews were videorecorded and audiorecorded, transcribed, and annotated. Teaching observation cycles were audio-recorded, transcribed, and annotated. Still photos from the teaching observation cycles were also collected. The task-based interviews were coded line-by-line for Fiona's use of processes and/or products by the research team. The coded instances were elaborated, categorized into the four process/product categories, and analyzed for emerging themes. Any disagreements were resolved by review of the data by the entire team. This procedure was repeated for the teaching transcripts, but included the coding and analysis of mathematical activity and pedagogical choices. After the initial coding and analyses, the team then compared Fiona's use of process and/or products in her personal mathematics with their use in her classroom mathematics.

Figure 1 illustrates parts of the Cube and Area tasks. In Cube, Fiona was asked to describe the pattern and determine the surface area and volume of the model shown in the left panel of Figure 1. In Area, Fiona was asked to describe the mathematical relationship between the sum of the area of the circles and the area of the equilateral triangle shown in the right panel of Figure 1 as the number of circles on the base increased. [One side of each equilateral triangle passes through the centers of the circles on that side and the endpoints of the same side lie on circle(s).]

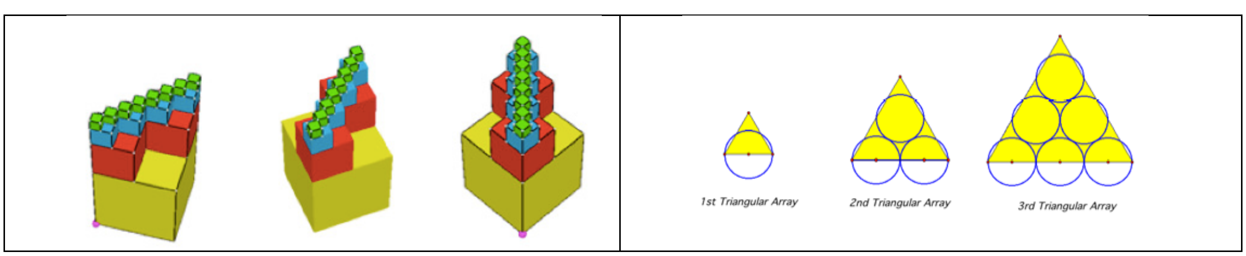

Figure 1: Some of the illustrations accompanying the cube and area tasks

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Results

Representing

Fiona tends to notice and pay attention to only select features of representations rather than accounting for the representation's complete set of relevant characteristics. In Area, when Fiona was given the graph of the function defined as the difference between the area of the triangle and the sum of the areas of the circles, she focused on the x-value of the minimum point of the function as invariant and interpreted it as the point at which the area of the triangle exceeds the sum of the area of the circles. She seemed to recognize that the graph of the differences required a change from negative to positive without recognizing that the negative-to-positive change captured by the minimum was a change in slope rather than a change in the output value of the function. She did not attend to the x-intercepts, the feature of the graph most relevant to the question she was asked, until she was specifically asked about them.

In her classroom mathematics, Fiona missed opportunities that might have engaged her students in linking different representations. For example, in student teaching, she purposefully did not interpret for students a graphical representation of derivative that appeared on an activity sheet. During the post-observation interview, Fiona ably linked this graphical representation to a symbolic representation of derivative but she stated that "they don't know that, and if I would explain it to them I would have confused them, I think endless amounts" as the reason for not discussing the graphical representation. She seemed to have made a conscious choice not to include this in her lesson. This might have been because her goal was only for students to be able to apply the limit definition of derivative in order to complete exercises and she thought that trying to develop further understanding of the limit definition was not worth confusing students.

In both her personal and classroom mathematics, Fiona often focuses on local features of representations and seems not to grasp the entire representation. This tendency of localization and inattention to connections is frequently observed in her personal and classroom mathematics and suggests that mathematics as an integrated system is not central to her conception of mathematics.

Justifying

In her personal mathematics, Fiona regularly makes initial mathematical claims for which she provides no or limited mathematical rationale. Fiona offers mathematical justification unprompted only when she recognizes an error and engages in correcting the error. Otherwise, Fiona justifies her mathematical claims only after she is prompted by the interviewer with questions such as, "How might you convince someone of your claim?" Moreover, when Fiona justifies by referencing properties of mathematical objects, she often fails to complete a valid mathematical argument. In these instances, Fiona often attends to one property of the mathematical object, but fails to attend to other relevant and necessary properties. In the Area interview, for example, Fiona engaged in justifying that the sum of the areas of the circles in an array is larger than the area of the triangle in the same array. She identified two differences in the symbolic representations of the two areas, but based this argument on one difference in the formulas (one area formula involved multiplying by π and the other involved dividing by the square root of three) without accounting for the other difference ($(x + (x – 1) + (x – 2) + … + 0)$ versus x^2) (see Figure 2).

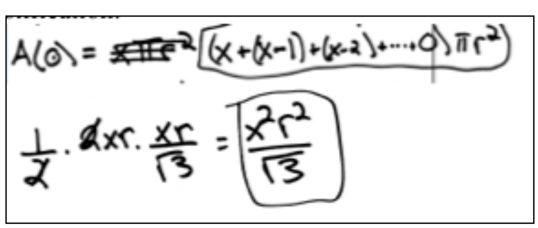

Figure 2: Fiona's representations of the sum of areas of the circles and the triangle area

In the context of Fiona's classroom mathematics, Fiona seldom engages in mathematical justification or asks students to justify. Across all three years of teaching, Fiona regularly misses opportunities to engage students in justifying. One of these instances occurred as Fiona taught rules for derivatives in student teaching. Fiona followed her presentation of the product rule with a presentation of the quotient rule. Even when a student pointed out the similarity between the product rule and the quotient rule, Fiona did not use the comment as a segue to recognizing and justifying that the quotient rule can be viewed as an instance of the product rule.

When Fiona engages in justifying or asks students to justify she often accepts superficial rationales. These superficial rationales were often rules or step-by-step procedures Fiona taught the students to use for a set of homework exercises. In first year teaching, for example, Fiona asked the students to justify the claim that 156 is the y-intercept of the equation $y = 78x + 156$. Fiona accepted a student response of "Because I know the equation is y equals $mx + b$ and then b is the y-intercept," echoing a fact that Fiona had taught during the previous class.

Instances of justifying in Fiona's personal mathematics and teaching seem to indicate that Fiona sees the role of justifying as verifying an assertion rather than as a critical process in her mathematics. Although Fiona ably demonstrates her ability to justify in her personal mathematics, she rarely does so without prompting. Often her justifications are invalid, because she attends to only some of the relevant features or properties of a mathematical object. In Fiona's classroom mathematics, she justifies or has students justify in the context of reviewing homework exercises or in-class examples. These justifications are usually superficial rationales or rules that do not provide mathematical connections.

Generalizing

In Fiona's personal mathematics, she generalizes but rarely uses generalizations. Although she generalizes, she does not tend to generalize without being prompted even when it seems reasonable to do so. For example, in Cube, Fiona was asked to find the volume and surface area of a stack of cubes. Fiona recognized that the volume of a cube in one particular layer was one-eighth the volume of a cube in the previous layer, but was hesitant to conclude that this was true for all layers and did not do so until she was prompted. Fiona's generalizations are not always correct, and these incorrect generalizations are often due to her focusing on a limited domain or set of examples and not accounting for all possibilities. For example, in Count when asked about a three-dimensional analogue of the circles situation shown in the right panel of Figure 1, Fiona generalized that no matter the size of a pyramid constructed of spheres, there are no interior spheres. She based this incorrect generalization on having examined only a single case.

In Fiona's classroom mathematics, there were many more instances of generalizations than generalizing. Similar to her personal mathematics, Fiona does not engage in generalizing when it seems that it would be appropriate to do so. For example, rather than giving students a single equation that can be used in various exercises, Fiona directed students to use three different equations for three different, but clearly related, cases: the equation $y/x = k$ to find the value of k, the equation $y = kx$ to find the value of x, and the equation $f = kd$ to find the value of k in an exercise involving Hooke's Law. Fiona discusses and implements activities that potentially provide students the opportunity to engage in generalizing. However, when she implements activities aimed at generalizing, she usually leads students to reach a generalization that she has predetermined rather than allowing for generalizations she has not anticipated. The theme in her teaching seems to be that generalizations are finished products, suggesting that she may consider the universe of possible generalizations as fixed and known. Also, Fiona states generalizations that are false, often based on a limited domain or set of examples. For example, when introducing a lesson on graphing lines, Fiona states the incorrect generalization, "There is a y-intercept and an x-intercept for every single line," not accounting for horizontal or vertical lines.

Although Fiona's personal mathematics did not make use of generalizations and her classroom mathematics contained almost no generalizing, Fiona's personal mathematics and her classroom mathematics have two main commonalities. First, Fiona often does not generalize when it seems as if it would be appropriate to do so. Second, Fiona often incorrectly generalizes or states incorrect generalizations because she is focusing on a limited domain or examples and is not accounting for all possibilities. The absence of further commonalities in Fiona's generalizing and use of generalizations may be attributed to the lack of the use of generalizations in her personal mathematics and the lack of generalizing in her classroom mathematics.

Defining

As with other processes, the process of defining does not seem to play a central role in either Fiona's personal mathematics or her classroom mathematics. We hardly ever observed her engaging in defining and most often we saw her engaged with the product of defining, namely definitions. In her personal mathematics and teaching, she tends to focus on elements of definitions rather than thinking about the definition as a whole. She seems to compartmentalize definitions and not to coordinate them in her teaching or in her personal mathematics. In the task-based interviews, she was presented with six ways in which people may talk about a parallelepiped (see Figure 3). She was asked which of the six descriptions are most similar to each other. If Fiona were to be thinking of these six statements as defining six separate objects, we would expect her to consider the mathematical entity each statement defines and to compare those entities. However, she chose to examine parts of each statement and to compare them to parts of other statements. For example, she stated that descriptions B and F are similar to each other because they both describe a six-sided polyhedron. However, she never endeavored to examine each description as a whole.

a. **A parallelepiped is a box with a lid.**
b. **A parallelepiped is a polyhedron with six faces that are parallelograms, which could be rectangles, squares or rhombuses but not trapezoids.**
c. **A parallelepiped is a cube whose sides are rectangles.**
d. **A parallelepiped is a prism the base of which is a parallelogram.**
e. **A parallelepiped is a hexahedron each face of which is a parallelogram.**
f. **A parallelepiped is a hexahedron with three pairs or parallel faces.**

Figure 3: Six ways people may talk about a parallelepiped

This tendency to focus on parts of a definition, rather than on the whole definition, is also reflected in her teaching. For instance, Fiona presented a definition of a vertex as "a point at which three or more faces meet." However, later on during the same lesson, Fiona introduced the phrase, "a vertex of a cone," offering a description: for a cone, "a curved surface connects the base to the vertex." One of her students pointed out that what she has labeled as a vertex is not actually a vertex, given the original definition of a vertex. Fiona agreed with the student without offering an explanation for her agreement. In the post-observation interview, Fiona explained that the student who asked the question is "very smart" and answering his question would just confuse the other students in the class.

The previous example involving the vertex also highlights how Fiona seems to view mathematics as a static and fixed body of knowledge, rather than something that can be discovered using the processes. If Fiona were to privilege a perspective of mathematics that encourages involvement in mathematical processes such as defining, we would have expected her to engage the student in a discussion of the mathematical properties of the two definitions of a vertex. This view is further illustrated in her first year of teaching when she presented her students with a definition of a *y*-intercept of a line. Fiona asked one of her students to read the textbook definition of a *y*-intercept, "The *y*-value of the point where the line crosses the *y*-axis." However, when Fiona repeated this definition and subsequently used it, she re-worded it as, "The *y*-intercept is the point where our line crosses the *y*-axis." Fiona seemed unaware of the change she made to the definition of a *y*-intercept, or she may not have seen a difference in the two definitions presented. Either possibility points to her seeing mathematics as comprised of static and disconnected pieces of information, and thus, if these pieces of information conflict with each other, she does not seem perturbed by it.

Conclusions

Two themes characterize Fiona's use of processes and actions on the products of those processes. These themes cut across several processes and are evidenced both in her personal mathematics and in her classroom mathematics.

Themes

The first theme is that, although Fiona has demonstrated competence in engaging in all four of the mathematical processes studied, the processes of generalizing, justifying, and defining are not central to how she engages in mathematics or how she engages students in mathematics. In her task-based interviews, Fiona generally engages in these processes only when prompted. Similarly, while teaching, she seldom engages in processes or requires that students do so even though we observed several occasions (e.g., students asking Fiona for justification or Fiona providing students with activities designed to lead to generalizing) in which it would have seemed reasonable to engage in processes. Fiona is far more likely to engage in actions on the products of processes than to engage in processes or to engage her students in those processes. The possible exception to this is the process of representing. In her personal mathematics, Fiona seems to use representing to help her in problem solving, often using one type of representation to create another and connecting representations to provide justifications. In her classroom, although Fiona occasionally directs students to use different representations in problem solving (e.g., directing students to draw a graph if they are struggling with writing an equation of a line or having them use geometric figures to generate tables of values in order to look for a generalization) she often misses opportunities to have students examine multiple representations even when it would seem to make sense to do so (e.g., not showing students a graph to explain a limit definition of derivative).

A second theme is that Fiona has a tendency, when working with processes and actions on the products of those processes, to focus on some features of a product or mathematical object and not attend to other relevant features. In her personal mathematics, many of Fiona's justifications are incorrect because she has not attended to all of the relevant characteristics of the object in question. In the Defining interview, Fiona incorrectly defines a particular set of polyhedra as having exactly one pair of parallel faces without recognizing that some of the polyhedra in the set have more than one pair of parallel faces. In her classroom mathematics, she uses the term vertex as having universal applicability and fails to distinguish between definitions of a vertex of a polyhedron and a vertex of a cone.

Conceptions of Mathematics

Much of the work to date on teachers' and students' conceptions of mathematics has been focused on describing and categorizing these conceptions. For example, Ernest (1988) categorized conceptions of mathematics into three broad categories: an instrumentalist view of mathematics as a set of unrelated but utilitarian rules and facts, a Platonist view of mathematics as a static body of knowledge, and a problem solving view of mathematics as dynamic and continually expanding. Lerman (as cited in Thompson, 1992) identified two different prevailing conceptions of mathematics: the absolutist perspective, that is, the perspective that "mathematics is based on universal, absolutist foundations" and the fallibilist perspective that "mathematics develops through conjectures, proofs, and refutations and is accepted as inherent in the discipline" (Thompson, 1992, p. 132).

There is growing evidence that teachers' conceptions of subject matter have an influence on their classroom instruction and there have been a few studies that provide evidence of the link between teachers' conceptions about mathematics and their instruction (e.g., Cross, 2009; Raymond, 1997; Thompson, 1984). It is evident that this influence is not direct or simple and results are inconsistent about how strong that influence may be.

Although Fiona did not speak directly about it, it is conceivable that Fiona's conception of mathematics could explain her approach to processes and actions on the products of those processes in both her personal mathematics and her classroom mathematics. The pervasiveness and consistency of the two themes we identified across the four processes and in both Fiona's personal and classroom mathematics seem to indicate that Fiona has a conception of the nature of mathematics that is not centered on the use of mathematical processes and does not require attention to connections and consistency. It seems likely that Fiona views mathematics as a fixed body of facts, ideas, and rules that are not necessarily or easily connected. Within such a concept of mathematics many of Fiona's actions make sense. For example, if mathematics is a fixed set of ideas then it is reasonable that Fiona does not encourage creative generalizing but chooses to focus attention on the generalization students are supposed to be learning. Fiona's apparent lack of attention to the fact that mathematics needs to be connected and coherent helps to explain why she seems not to notice or be concerned about contradictory definitions of vertex. Meanwhile, a view that mathematics is fixed and static may explain why, when a student points out the discrepancy, she might find it adequate simply to tell students that "it's part of the definition."

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