A RETROSPECTIVE ANALYSIS OF STUDENTS' THINKING ABOUT VOLUME MEASUREMENT ACROSS GRADES 2–5

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This paper describes a retrospective analysis of data collected during a 4-year longitudinal study on children's thinking about measurement through a teaching experiment methodology. It focuses on results from individual interviews with two students on volume measurement. Data analysis was guided by Sarama and Clements' (2009) learning trajectory on volume measurement. Results indicate that (1) both students progressed through levels of the learning trajectory during the study, (2) different representations of 3-D objects (e.g., physical objects, cubes, pictorial representations) influenced their strategies, and (3) their individually constructed definitions for the term "volume" affected their decisions in volume comparisons.

Keywords: Measurement; Learning Trajectories; Geometry and Geometrical and Spatial Thinking; Elementary School Education

Background and Rationale

Measurement can bridge two critical domains of mathematics, geometry and number, as well as provide conceptual support to those domains (Clements & Sarama, 2007). Research has shown that children have difficulty in fully grasping the concept of volume measurement (Battista & Clements, 1996; Ben-Haim, Lappan, & Houang, 1985; Enochs & Gabel, 1984). In measurement contexts, including volume, many children apply formulas to get the answers without understanding the meaning of these formulas (Clements & Battista, 1992). Nation-wide assessments also revealed student difficulties in solving volume measurement problems. The 1977–1978 results from the National Assessment of Educational Progress (NAEP) showed that less than 50 percent of the students at grade 5 through 8 were able to answer questions correctly about the volume measurement of three dimensional (cube) arrays (Ben-Haim et al., 1985). Students' errors stemmed mostly from counting the number of visible faces of cubes shown, counting the number of visible cubes in the diagram, estimating the number of faces of cubes shown in a given diagram, and double counting cubes (Battista & Clements, 1996; Ben-Haim et al., 1985). The researchers stressed that many students were unable to enumerate the cubes correctly in such an array. Overall, these studies showed that students could not correctly solve volume measurement tasks because they were (a) not correctly applying the volume formula, (b) not correctly counting the number of cubes in 3-D arrays, or (c) confounding volume and surface area measurement. Although the current research tells us much about children's thinking in volume measurement, missing is longitudinal work showing how students' thinking about volume measurement grows throughout their development and with the instruction they receive.

Theoretical Framework

The theoretical perspective guiding this study is described by the framework of *hierarchic interactionalism,* which indicates "the influence and interaction of global and local [domain specific] cognitive levels and the interactions of innate competencies, internal resources and experience" (Clements & Sarama, 2007, p. 464). Of the 12 tenets of *hierarchic interactionalism*, the learning trajectories tenet (Sarama & Clements, 2009) is the most germane to this report. It guided the design of the longitudinal study and informed the development of instructional tasks. A hypothetical learning trajectory consists of three components: a learning goal, a likely path for learning as students progress through levels of thinking, and the instruction that guides students along the path (Sarama & Clements, 2009). Furthermore, the learning trajectory for volume measurement was utilized as the data analysis tool for this study.

Sarama and Clements (2009) asserted that students' understanding of volume measurement gradually improves with the instruction they receive in addition to their natural development. They defined the volume measurement trajectory through eight levels. According to the trajectory, children initially focus on external aspects of arrays as sets of faces. Later, students develop an appreciation of the internal structure of 3-D arrays. They gradually become capable of counting the number of cubes contained in objects, one by one or in a pattern of rows and columns and layers. This level is called Primitive 3-D Array Counter (PAC). The next level, Capacity Relater and Repeater (CRR), focuses more on volume as capacity. At the CRR level, a student "fills a container repeatedly with a unit and counting how many. With teaching, [a student] understands that fewer larger than smaller objects or units will be needed to fill a given container" (Sarama & Clements, p. 307). The next level, Partial 3-D Structurer (PS), describes student counting in terms of rows or columns (or units of units) of a solid built with unit cubes. The next more complex level of volume involves thinking in terms of layers of unit cubes and is called 3-D Row and Column Structurer (VRCS). The highest level described in the volume measurement learning trajectory is 3-D Array Structurer (AR). At this level, students can mentally de/compose 3-D arrays into layers. These levels are used to describe student's thinking for a particular task or teaching episode rather than to define the student's overall thinking about volume measurement.

Purpose

The aim of this study was to investigate students' thinking in volume measurement over a four-year period within the context of a teaching experiment.

Research Question 1: How do students develop coherent knowledge and integrated strategies for volume measurement across Grade 2 through Grade 5?

Research Question 2: How are students' abilities for spatial thinking, algebraic reasoning, or proportional reasoning related to their measurement knowledge and strategies?

Method

The sample for this report consisted of two children (Ryan and Owen) from a Midwestern public school. Each student represents just one of seven case studies within a four-year longitudinal study investigating children's thinking and learning across length, area, and volume for Grades 2–5. Instructional tasks were developed within the context of a teaching experiment (Steffe & Thompson, 2000). The teaching experiment consisted of a series of teaching episodes for which the research team generated a set of tasks and predictions for student responses and then later checked student responses against these predictions. Each teaching episode was an individual, semi-structured interview, which lasted 15 to 40 minutes. The interview tasks were informed by the learning trajectory for volume measurement (Sarama & Clements, 2009). Before the first teaching episode, an initial assessment was administered in the form of a clinical interview in which the interview tasks were posed without feedback or instruction. All interviews, including the initial assessments and teaching episodes, were videotaped, transcribed, and analyzed by a group of researchers, the authors.

During the four-year teaching experiment, Owen encountered 30 volume measurement tasks within 12 interviews, and Ryan encountered 30 volume measurement tasks within 11 interviews when they were in third, fourth and fifth grades in addition to their initial assessment interview during second grade. The teaching episode tasks represented volume with a variety of objects (e.g., physical objects, cubes, pictorial representations). Additionally, the tasks required a variety of actions: filling a container with water, packing a box with the unit cubes, building a prism with unit cubes, and finding the volume given only linear measurements. Some of the tasks required students to draw 3-D objects.

Results and Discussion

Five themes emerged from the analysis of the data for both students: (a) finding volume of rectangular prisms, (b) relating the size and the number of units, (c) visualizing representations of the 3-D objects, (d) flexible unit sense, and (e) fractions and volume measurement. Due to space limitations, only the first three themes will be presented in this paper.

Owen's Thinking on Volume Measurement

a) Finding volume of rectangular prisms. In the second grade spring semester—during his initial assessment, Owen was asked to determine the number of cubes needed to make a 3 in × 2 in × 2 in prism presented as a physical object with the individual cubes displayed. While holding the prism and showing the interviewer what he counted, Owen said, "12…because there is 3 here [showing one whole surface of one face] …and 3 here [showing another whole surface of one face]." Later, he changed his answer to 6 (correct). While determining the number of cubes, Owen mentally constructed and counted layers (composite units) of 6 in the figure. However, when asked how many cubes altogether would be needed to fill the partially filled box of 3 units × 3 units × 4 units (Figure 1), he looked at the figure and said "about 35" without any apparent strategy.

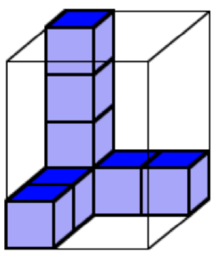

Figure 1

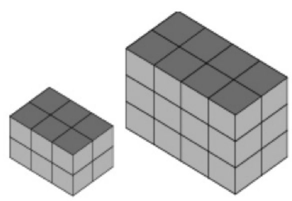

Figure 2

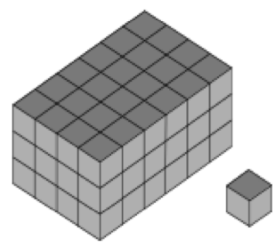

Figure 3

Approximately 12 months later, in the third grade spring semester, Owen was asked to compare the volume of the two prisms in Figure 2. Although he reported a correct additive comparison of 12 more cubes would be needed to build the larger prism, to determine how many blocks would be needed to make the smaller prism (3 units × 2 units × 2 units), he incorrectly answered 16. When asked to build the first figure with the actual cubes, Owen built the figure by using an appropriate row structuring strategy, with 12 cubes. He made two separate rows of 3, placed them next to each other, then explained that there were two layers of 6 in the figure, and changed his answer from 16 to 12. In response to an extension of the same task, Owen said that he would need 36 cubes to build the second figure (4 units × 2 units × 3 units) shown on the paper. Pointing to one of the lateral faces of the figure, he explained 6 plus 6 is 12 and then pointed to the front face and lateral face respectively and said "another 12." Owen appeared to have attended to the surface area of the vertical faces only. When asked to build the figure with cubes to check his answer, he constructed rows and layers correctly and changed his answer to 24.

Throughout the interview, Owen did not initially determine the number of cubes correctly. Instead he first attended to the surface area or counted the squares on the lateral faces. These actions are consistent with the PAC level. On the other hand, when asked to build, Owen correctly resolved the tasks at the PS level by skip-counting and thinking about volume in an organized way; he was adding the number of units in rows.

Approximately eleven months later, in the spring semester of fourth grade, Owen was asked to compare the volume of a rectangular prism to a unit cube (Figure 3). Owen stated that one cube on the side was one of the cubes in the rectangular prism and counted the number of cubes on the bottom. He said that 5 times 4 was 20 and 20 times 3 was 60. In this task, he saw that there were 3 horizontal layers in the solid. However, he calculated the number of cubes in each layer incorrectly; he counted the length as 5 instead of 6, likely in an attempt to avoid double counting a row. Nevertheless, this showed that he could think at the

VRCS level by counting layers of units multiplicatively. Later, in the same interview, Owen was given the same task again (see Fig. 1) and asked how many cubes would be needed to fill the outlined box. By attempting to count the cubes individually Owen gave an incorrect answer of 27. He did not use layers or multiplicative thinking although he used the strategy of structuring in terms of layers for the previous tasks of the same interview. Owen may have failed to resolve this task successfully because of the high visualization demand of the task.

Approximately eight months later, in the first interview of fifth grade, Owen was given a solid box (4 in × 3 in × 3 in), which did not have any unit indication on it and a collection of 1-inch cubes and was asked how many small boxes it would take to fill the big box (Figure 4). Owen struggled in the beginning and gave an incorrect answer of 12. After building one vertical layer of 4 in × 3 in × 1 in, which aligned to one face of the box, he changed his answer. He found the number of cubes in one layer, 12, and thought there were "3 rows [layers]" so 36. In this instance, Owen showed the VRCS level of thinking as evidenced by his tendency to think about prisms in terms of layers built from cubes.

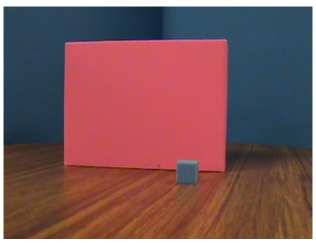

Figure 4

Figure 5

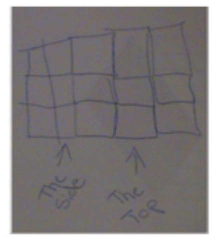

Figure 6

One of the tasks for the second interview of the semester was about finding the volume of the room compared to a cubic meter and a cubic decimeter (Figure 5). After iterating a meter stick across the floor of the room, 6 times for the length and 3 times for the width of the base of the room, Owen reasoned correctly that 18 cubic meters would fit in the bottom layer of the room. He said that there would be "3 of those [layers] going high…so 18 times 3." He concluded that therefore, 54 cubic meters would fit into the room. He also measured all three dimensions of the cubic meter and one dimension of the cubic decimeter block with a meter stick and found that 10 cubic decimeters would fit along each edge of the cubic meter. Thus, he multiplied 10 times 10 times 10 and found a product of 1000 to represent the number of cubic decimeters in one cubic meter. In order to determine the number of cubic decimeters that would fit into the room, he multiplied 54 by 1000. Owen found approximate values for the measurement of the dimensions of the room (actual room size: 7 m × 4 m × 3 m). The student applied multiplication and applied AR level strategies to resolve this task by using only the linear measurement of the 3-D prisms, by building and manipulating composite units of cubic decimeters as well as cubic meters, and by mentally decomposing arrays into layers, rows, and columns.

b) Relating size and number of units. Owen encountered a number of tasks requiring him to relate the size and number of units. Starting in third grade, as suggested in the volume measurement trajectory CRR level, he was aware that different units would give a different volume measurement and he recognized an inverse relationship between the size and number of units.

c) Visualizing representations of the 3-D objects. In the spring semester of fourth grade, Owen was asked to draw something that was 3 times as big as a 1-unit cube. Owen drew three squares in a 3×1 form. Next, he was asked to draw a picture of a solid three times the volume of a 3×2×1 solid. He created the drawing shown in Figure 6. With an aerial perspective, he could visualize the new solid from a top view. In the follow up interview, while looking at his old drawing, Owen explained that there would be 2 layers of 9 so that there would be 18 cubes. The figures Owen drew did not have 3-D perspective; however, he could use his own representations to determine the number of cubes in the figures.

Ryan's Thinking on Volume Measurement

a) Finding volume of rectangular prisms. In the second grade spring semester initial assessment, Ryan was asked how many cubes altogether would be needed to fill the partially filled box of dimensions 3 units × 3 units × 4 units (Figure 1). Ryan drew line segments extending the edge lines of the cubes and gave an incorrect answer of "18."

In the third grade spring semester, when asked how many of these cubes were necessary to make the smaller shape in Figure 2, Ryan gave the answer 25 (incorrect) and explained that he counted 4 for each lateral side, 6 for the front side, and 6 for the ceiling. He did not count the cubes on the base and also added the numbers incorrectly. Ryan was asked to build the physical representation of the shape with actual unit cubes. After building he noticed his mistake and said, "I was counting them wrong…I was counting the sides…I should have been counting cubes." He also took the cube in the top corner and said that he was counting that cube as two although he should have counted it as one. His final answer of 12 was correct after being allowed to build the structure with unit cubes. Ryan's realization of his incorrect strategy might have influenced his volume definition throughout the task, which changed to "how many cubes there are." In the next task of the same interview, while finding the number of cubes in the second figure of dimensions 4 units × 3 units × 2 units (Figure 2), Ryan curtailed his previous strategy of counting the faces of cubes. However, he still answered incorrectly (28 cubes) because he only counted the visible faces of some cubes. When he was asked to build the physical representation of the figure, he built the actual figure row by row and reached the correct answer. Ryan's responses and strategies, while resolving these tasks, were consistent with the PAC level. Specifically, he initially counted the outer faces of cubes. On the other hand, while building the shape with the actual cubes, Ryan could think more systematically in terms of columns as exemplified in the PS level. Ryan used a higher level of strategy while using 3-D physical objects than when resolving the task with the figural representations on the paper.

Seven months later, in the fall semester of fourth grade, Ryan was shown the actual cubes and container in Figure 7 and asked how many blocks would be needed to fill the container. Ryan counted by fives up to 20 by showing the bottom rows of 5 and said that there would be about 20 blocks. Then he counted by 20s up to 100 reporting a correct answer. When the same type of question was asked for a larger container (8 in × 10 in × 5 in) and with missing cubes in the first layer (Figure 8), Ryan failed to give a correct answer for the number of blocks needed to fill the whole container. In order to prompt the student, the interviewer took the extended rows and columns so the shape was changed back to its original version, and reminded Ryan that he said 100 blocks would fill the previous container. Next, the interviewer added 4 more cubes to the side of 4 and repeated the question. Ryan explained that the complete part had 4, the missing part also had 4 and explained that if they filled the missing part, 40 would go in the first layer, 80 in two layers, and so forth; he added composite units of 40s until he reached a correct response of 200. In the first task of the interview, Ryan counted first row by row and after finding the number of cubes for one layer, iterated the numbers by adding for each layer. However, he could not implement the same strategy properly when both rows and columns were missing on the bottom layer or when the numbers got bigger than 5 or 6 in a row.

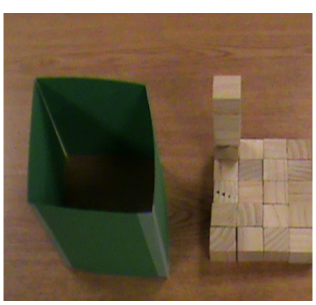

Figure 7

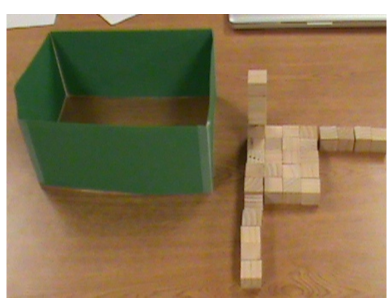

Figure 8

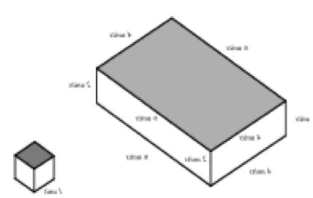

Figure 9

Although Ryan showed VRCS level strategy of counting composite units and then counting layers by skip counting in the first task, he did not use this strategy when the mental structuring demand increased as in the later task. Once he knew how many cubes were in one layer, he used repeated addition to add up the blocks in all the layers. This strategy is a typical VRCS strategy in the learning trajectory.

Eight months later, in the spring semester of fourth grade, Ryan was shown the task represented in Figure 9 and was asked, "If this cube has a volume of one, what is the volume of this solid? How many cubes would it take to fill the box?" Ryan found the volume by adding all of the number labels given for the edges. This task representation did not elicit his thinking clearly whereas the previous task was useful to probe his thinking about structuring in volume measurement.

Approximately five months later, in the first interview of fifth grade, Ryan was given a rectangular prism (4 in × 3 in × 3 in), which was wrapped with paper, and a 1-inch cube (Figure 4). He was asked how many of the cubes it would take to fill the box. By iterating the cube along one side of the prism, Ryan said "it is about 36….1,2,3,4." He pointed to a row of 4 on the bottom of the lateral face by iterating the cube and continued to iterate vertically by counting "4, 8, 12." Then he continued to count by pointing with his finger on the upper edge of the box vocalizing 12, 24, 36. Later, when given a centimeter cube and asked how many of those cubes would fit into prism, Ryan made a composite unit of 8-centimeter cubes to represent each 1-inch cube. Next, he reasoned correctly and multiplied 36 by 8 and gave an answer of 288. Ryan showed VRCS level of thinking by mentally decomposing a 3-D array into layers. Additionally, his strategy of building a composite unit of 8-centimeter cubes was consistent with AR level.

b) Relating size and number of units. In the spring semester of third grade, Ryan's volume definition was, "the number of cubes in a shape." In order to determine whether he was able to relate the unit size and volume, he was posed a task requiring him to compare the volume of a 4 in × 2 in × 2 in rectangular wooden prism with a 4 cm × 3 cm × 2 cm rectangular yellow plastic prism. He was told, "Another person compared the figures and thought that since the yellow prism has 24 cubes and the wooden one has 16, the yellow figure has a larger volume. Do you agree?" Ryan thought, by his definition of volume, that the volume of the block made of centimeter cubes was greater even though he articulated that two wooden cubes are just like 24-centimeter cubes. Similarly, in the spring semester of the fifth grade, he was given a task relating different unit sizes and the number of units in volume measurement. When asked which objects would melt into more water, Ryan thought that the collection of smaller cubes would melt into more water because there were more of them even though he noticed the difference in the unit size. He stated that they would need four of the larger cubes to "make" the six of the smaller cubes. We suggest that Ryan's volume definition had not substantially changed through third and fourth grade, nor had he conceptualized volume as the amount of space an object would take up.

c) Visualizing representations of the 3-D objects. In the spring semester of fifth grade, Ryan was asked to make a drawing on paper of a 1-inch cube that he was handed, and second to draw a picture of 3 in × 2 in × 2 in figure shown in a two-point perspective drawing with shading. He drew a square and called it a cube. He struggled to copy the 2-D drawing of the 3-D image.

Conclusions and Implications

According to the results, when finding the volume of rectangular prisms, both Owen and Ryan demonstrated abilities in each of the levels of PAC, PS, VRCS, and finally AR. However, both students also employed lower level strategies for some tasks.

Consistent with prior research, (e.g., Battista & Clements, 1996; Ben-Haim et al., 1985), initially, both students had the tendency to count the outer faces of the cubes instead of the number of cubes, demonstrating PAC level strategies. After one or two semesters, they could see the row, column, and layer structures in the rectangular prisms and count the number of cubes by creating composite units (row, column, layers). In later semesters, both students used multiplication as repeated addition while thinking in terms of rows, columns, and layers made of unit cubes.

Ryan mostly used multiplication when the figures were represented physically but not pictorially. These findings suggest that it is important to provide a variety of representational forms of 3-D objects in volume measurement tasks in order to help students solidify the meaning of multiplication in volume measurement. In addition, the formula for the volume of a rectangular prism may not make sense to students without understanding why they need to multiply the linear dimensions (Clements & Battista 1992).

In spring semester of fourth grade, although Ryan and Owen struggled to determine the volume of rectangular prisms without any grid on them, they were able to resolve some of the tasks when prompted to use a 3-D unit. For example, both students could resolve the task represented in Figure 4 by using the unit cube given. Ryan iterated one unit cube along the edges of the cube to find the number of unit cubes in each row, column, and layer. Owen was prompted with multiple cubes aligned as a vertical layer of the prism. Therefore, giving a 3-D solid unit was helpful for both students.

Students' thinking differed according to the physical versus pictorial representations provided in the tasks. When both students were posed tasks requiring them to imagine rows, columns, and layers, and some aspects of the figure were obscured, the students sometimes gave incorrect answers and used incorrect strategies. For the task represented in Figure 1, Owen struggled to visualize the complete rows, columns and layers if the box was full of cubes. Similarly, Ryan could not resolve the task (Figure 8) when there were hidden rows and columns on the bottom layer, as the visualization demand was higher.

Based on students' responses, it can be claimed that students' strategies about relating the unit size and the total number of units in the objects were influenced by their volume definition. For Ryan, who defined volume as, "the number of cubes in a shape," the volume of the block made of more (centimeter) cubes was greater than the one made of fewer inch cubes even though he noticed the difference in the unit size. Therefore, while preparing the instructional materials for students, using different size units and comparing the volume of objects with those units should be considered, especially for the students who have a misconception that the volume is the number of units instead of the amount of 3-D space an object takes up.

Moreover, both students had difficulties in drawing or copying 3-D figures made of cubes on paper. Although they could calculate the volume of those objects, their drawings did not correctly represent the actual figures. We claim that Owen could interpret his own drawing and that he could hold a correct mental representation in his mind for the object. This showed that the students' representations might be identified as incorrect, even though they have correct mental representations for the figures in their minds. Students should be given opportunities to draw representations and at the same time given opportunities to articulate what they are seeing from the pictures. This might let teachers understand how students think and visualize the 3-D objects.

Lastly, on most of the tasks requiring students to interpret 2-D drawings or pictures of 3-D figures, students resolved the task if they were allowed to build the figures with the actual unit cubes. Building the figures with cubes apparently helped students identify their mistakes and change their strategy to count in terms of rows, columns, and layers. Therefore, students should be given opportunities to build the shapes, which are shown on paper.

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