THE USE OF CAS IN THE SIMPLIFICATION OF RATIONAL EXPRESSIONS AND EMERGING PAPER-AND-PENCIL TECHNIQUES

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In this paper we analyze and discuss students' performance in a CAS environment related to the simplification of rational expressions. Results indicate that if students have more initial paper-and-pencil techniques, the CAS environment spurs them to deeper theoretical reflections than for students who have fewer techniques.

Keywords: Task-Technique-Theory; CAS; Rational Expression

Background

In the last few years, an area of research interest in mathematics education has developed that deals with the influence of CAS technology in students' algebraic thinking. Thomas, Monaghan, and Pierce (2004), for example, have identified some crucial questions when considering the use of CAS in the learning of algebra: "How does the use of CAS influence student conceptualization? How does the way students work on tasks by hand inform their work in a CAS environment and vice versa?" (p. 166). These paramount questions and those arising from other recent studies (e.g., Kieran & Drijvers, 2006; Hitt & Kieran, 2009; Guzmán, Kieran, & Martínez, 2010, 2011) have driven our interest in this area. In particular, these studies and others have suggested the importance of the technical aspect in algebra learning in CAS environments.

Researchers such as Kieran and Drijvers (2006) have indicated that the use of CAS promotes conceptual understanding if the technical aspect of algebra is taken into account; these researchers have shown specifically that technical and theoretical aspects of algebra co-emerge in students' thinking. In this sense, and related with the simplification of rational expressions, Guzmán, Kieran, and Martínez (2010, 2011) have shown the epistemic role of the use of CAS when students confront their CAS work with their paper-and-pencil work. These studies are related to the transformational activity of algebra (Kieran, 2004)—a characterization of algebra in which the importance of technique acquires relevance in the sense that, within transformational activity, conceptual understanding can come with technique.

Guzmán, Kieran, and Martínez (2010, 2011) have shown that the use of CAS provoked spontaneous theoretical reflections in students, which allowed them to think of new techniques to simplify rational expressions. The use of CAS promoted a change in the students' technique for simplifying rational expressions whose denominator is a binomial (from canceling "literal components" that were repeated in both numerator and denominator to using the polynomial division algorithm). This epistemic role played by the CAS occurred in students whose initial technique was "cancelling literal components," but for whom the notion of cancelling "common factors" and dividing polynomials was absent. Based on our previous studies (Guzmán, Kieran, & Martínez, 2010, 2011), one can therefore ask the following question: What is the role of CAS in students' algebraic thinking if they already have as initial techniques "canceling literal components" and the "long division of polynomials" for simplifying rational algebraic expressions? Does CAS promote other techniques and theories? This paper will deal with this issue.

Theoretical Framework

The Task-Technique-Theory perspective, which is part of the instrumental approach to tool use, has been proposed as a framework for analyzing the processes of teaching and learning in a CAS context (e.g., Artigue, 2002; Lagrange, 2003). This approach encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). There are two directions within the instrumental approach: one in line with the cognitive ergonomics framework, and the other in line with the anthropological theory of didactics. In the former, the focus, according to Drijvers and Trouche (2008), is the development of mental schemes within the process of instrumental genesis. Within this direction, an essential point is the distinction between artifact and instrument.

In line with the anthropological direction, researchers such as Artigue (2002) and Lagrange (2003, 2005) focus on the techniques that students develop while using technology. According to Chevallard (1999), mathematical objects emerge in a system of practices (*praxeologies*) that are characterized by four components: *task*, in which the object is embedded (and expressed in terms of verbs); *technique*, used to solve the task; *technology*, the discourse that explains and justifies the technique; and *theory*, the discourse that provides the structural basis for the technology.

Artigue (2002) and her colleagues have reduced Chevallard's four components to three: *Task, Technique,* and *Theory*, where the term *Theory* combines Chevallard's *technology* and *theory* components. Within this (Task-Technique-Theory) theoretical framework a *technique* is a complex assembly of reasoning and routine work and has both pragmatic and epistemic values (Artigue, 2002). According to Lagrange (2003), technique is a way of doing a task and it plays a pragmatic role (in the sense of accomplishing the task) and an epistemic role. With regard to the epistemic value of technique, Lagrange (2003) has argued that technique plays an epistemic role in that it contributes to an understanding of the mathematical object [in this case the rational expression and its simplified form] that it handles, during its elaboration. Technique also promotes conceptual reflection when the technique is compared with other techniques and when discussed with regard to consistency (p. 271).

According to Lagrange (2005), the consistency and effectiveness of the technique are discussed in the theoretical level; mathematical concepts and properties and a specific language appear. This epistemic value of technique is crucial in studying students' conceptual reflections within a CAS environment. We took into account this Task-Technique-Theory (T-T-T) framework in the designing of the Activity related to the task "simplifying rational expressions," in the conducting of the interview interventions, and in the analysis of the data that were collected.

Unfolding of the Study

In this paper we report and discuss the data of the first two of four Activities designed for a wider research study on a Technical-Theoretical approach in the construction of algebraic knowledge in a CAS environment.

The Design of the Activity

Hitt and Kieran (2009) have pointed out that when taking into account the transformational activity of algebra it is important that the design of the Activity promote the articulation between techniques and theory construction. Since we adopted the T-T-T framework for carrying out the study, the Activities were designed so that technical and theoretical questions were central. We wanted students to have the opportunity to reflect on both technical and theoretical aspects throughout the Activity that was embedded in a CAS environment. It is important to mention here that both paper-and-pencil work and CAS work were intertwined within the Activity. In addition, in this study we use the term task as is defined in the T-T-T framework. As Kieran and Saldanha (2008) state, the Activity is a set of questions related to a central task, in this case the "simplification of rational expressions." In the study, we developed four Activities, each one related to different aspects of the simplification of rational expressions. In this paper we report only the results of the first two Activities, which both involved paper-and-pencil work and CAS work, both with technical and theoretical questions.

Population

This report focuses in the work of one team (two students); the full study included seven teams (two students each team). The participants were 10th grade students (15 years old) in a Mexican public school. The selection of the students was made by their mathematics teacher. None of the students were accustomed to using CAS calculators; consequently, at the outset of the study, all the students received some basic training from the interviewer-researcher on how to use the TI-Voyage 200 calculator for basic symbol manipulation (how to introduce algebraic expressions, the use of the *Solve*, *Expand* and *Factor* commands, the use of the *Enter key* and the use of the "*equal sign*").

Implementation of the Study

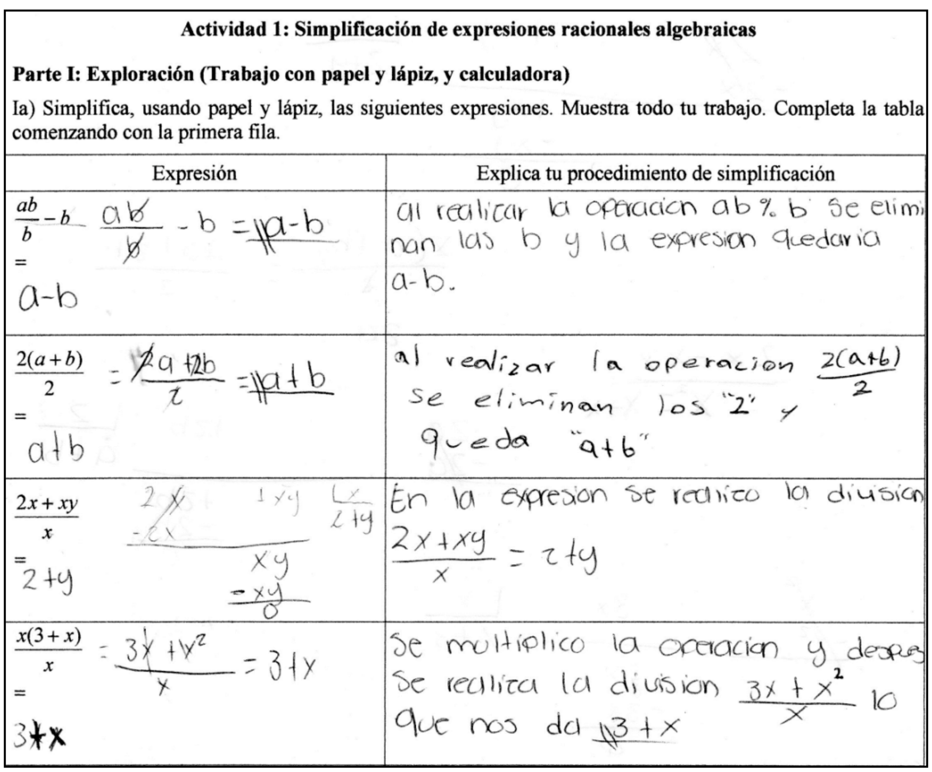
The data collection was carried out by means of interviews conducted by the researcher. Students worked in pairs; each work session lasted between two and three hours (for each Activity). Each team of two students had a set of printed Activity sheets as well as a TI-Voyage 200 calculator. Every interview was audio and video-recorded so as to register the students' performance during the sessions. So, our data sources included the audio and video recordings, the written Activity sheets, and the researcher's field notes.

Analysis and Discussion of Data

In this paper we analyse and discuss the work of one team. The team was chosen for this report because these students (we will call each of them Student A and Student B) used two techniques to carry out the task in the first Activity: Cancelling numbers or literal symbols that are repeated in the numerator and denominator of the rational expression, and at other times applying the long division technique. So the performance of these students fits the question that we try to respond to in this paper. The following analysis and discussion is restricted only to the first two of the designed Activities.

The Paper-and-Pencil Technique and Theory

As was mentioned before, in Activity 1, for those expressions that involved a monomial in the denominator, these students "simplified" the given rational expressions by using two techniques. One technique was cancelling the numbers or literals symbols that were repeated or common to the numerator and denominator. The following Figure 1 illustrates their paper-and-pencil work.



Actividad 1: Simplificación de expresiones racionales algebraicas

Parte I: Exploración (Trabajo con papel y lápiz, y calculadora)

Ia) Simplifica, usando papel y lápiz, las siguientes expresiones. Muestra todo tu trabajo. Completa la tabla comenzando con la primera fila.

Expresión	Explica tu procedimiento de simplificación
$\frac{ab}{b}-b$ $\frac{a\cancel{b}}{\cancel{b}}-b = a-b$ = $a-b$	al realizar la operacion ab % b se eliminan las b y la expresion quedaria a-b.
$\frac{2(a+b)}{2}$ $= \frac{\cancel{2}a+2b}{2} = a+b$ = $a+b$	al realizar la operacion $\frac{2(a+b)}{2}$ se eliminan los "2" y queda "a+b"
$\frac{2x+xy}{x}$ = $2+y$	En la expresion se realizo la division $\frac{2x+xy}{x} = 2+y$
$\frac{x(3+x)}{x}$ $= \frac{3\cancel{x}+x^2}{x} = 3+x$ = $3+x$	Se multiplico la operacion y despues se realiza la division $\frac{3x+x^2}{x}$ lo que nos da $3+x$

Figure 1: Students' paper-and-pencil work

In a first moment, the performance of these students was similar to that of others reported in an earlier pilot study in Guzmán, Kieran, and Martínez (2011). Students first expanded the expressions, and after that, they cancelled out the repeated elements in both the numerator and denominator. This technique works if the numerator is a binomial and the denominator is a monomial that is common to both terms of the binomial. The other paper-and-pencil technique that one can see in Figure 1 is the long division algorithm for polynomials. The explanations given by the students of these two techniques were more a description of what they did rather than a theoretical discourse. For instance, for the second expression (see Figure 1) they wrote: "When carrying out the operation… the 2's are cancelled and you are only left with $a+b$." For the third expression in Figure 1, their explanation included the terminology of *dividing*.

When the students were faced with expressions whose numerators and denominators were both binomials, they again used the techniques described above. Sometimes they used the long division technique and other times the "cancelling technique." As a result of using this latter technique applied to these kinds of expressions, they made well-known errors (Matz, 1980), that is, they applied the "cancelling technique" no matter whether the number or literal symbol they cancelled out was a common factor of both numerator and denominator or not (see Figure 2).

$\frac{4x+4y}{x+y} = 4$	$4y \quad +4y \quad \lfloor x+y$ $-4x \quad -4y \quad 4$ $4(\quad 0$	Se realiza la division $\frac{4x+4y}{x+y}$ / Se elimina "x" y se elimina 4y
$\frac{3x+4y}{x+y} = 7$	$\frac{3\cancel{x}+4\cancel{y}}{\cancel{x}+\cancel{y}} = 3+4$	al realizar la operacion $\frac{4x+4y}{x+y}$ se eliminan las "x" con las "x" y las "y" con las "y" = 3+4=7

Figure 2: Students' paper-and-pencil work on binomial over binomial expressions

The CAS Work (a First Theoretical Reflection)

Once students confronted their paper-and-pencil results with the CAS results, a theoretical reflection based on their long polynomial division technique emerged. At this point we can see that using a technique is not just a routine work, just as Artigue (2002) has mentioned. The performance of these students fits the results obtained in a previous phase (the pilot study) of the research (see Guzmán, Kieran, & Martínez, 2011). In this main study, the same kind of theoretical reflection was provoked by the use of CAS (see Figure 3).

Ii) Para los casos en que tu simplificación de Ia) no coinciden con los dados por la calculadora en Id), utiliza tu conjetura de Ig) (y que verificaste en Ih)) para simplificar de nuevo tales expresiones.		
Expresión para la cual tu resultado de Ia) no coincide con el dado por la calculadora en Id)	Simplificación de acuerdo con tu conjetura de Ig) (lápiz y papel)	¿En qué es diferente tu procedimiento de simplificación (segunda columna) respecto de tu procedimiento utilizado en Ia)?
$\frac{3x+4y}{x+y}$	$3\cancel{x} \quad +4\cancel{y} \quad \lfloor x+y$ $-\cancel{3x} \quad -\cancel{3y} \quad 3$ y El residuo no es cero lo que quiere decir que no se puede simplificar.	En que en Ia solo eliminamos los factores comunes y sumar los coeficientes. y en este caso hicimos la division.

Figure 3: Students' reflection based on their CAS work

In this part of the Activity they wrote (see the second column of Figure 3): “the remainder is not zero; that means that the expression cannot be simplified.” As reported in Guzmán, Kieran, and Martínez (2011), we consider this kind of discourse to be a spontaneous theoretical reflection. In the third column of Figure 3, they included terminology of *common factors*. However, because of their previous work, we can say that they did not really understand this aspect (common factors); for them, all numbers or literals repeated in the numerator and denominator are *common factors*. In Activity 2, when these students had the opportunity to explore other cases, the use of CAS played an important role regarding the idea of *common factors* and making this idea more mathematically clear.

Second Theoretical Reflection Based on the CAS Technique

After the first theoretical reflection emerged, the students used their long division technique in order to explain the CAS results each time they found discrepancies between their paper-and-pencil work and their CAS work. Figure 4 illustrates this.

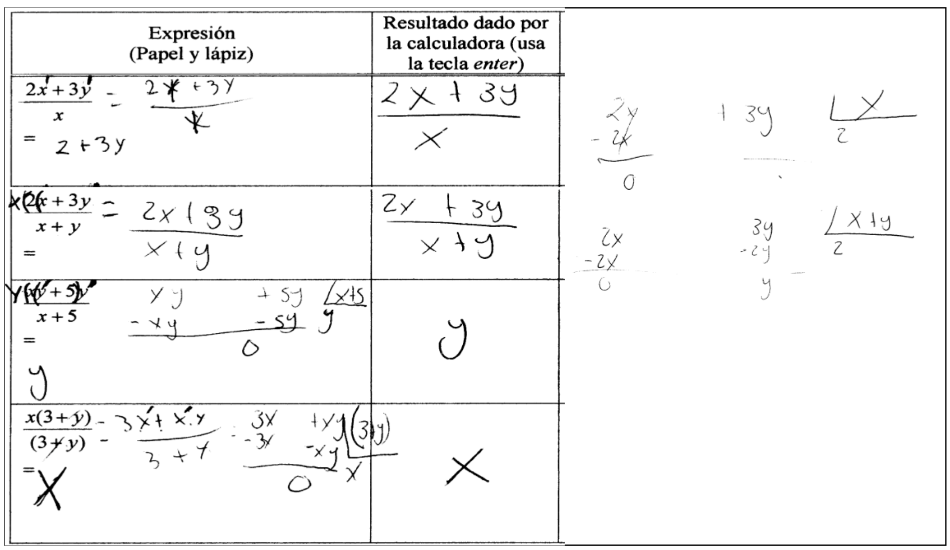


Figure 4: Use of long division technique in order to explain some CAS results

After the students had used CAS, their explanations (based on their theory of the remainder of the long division of polynomials algorithm) for simplifying expressions whose denominator is a monomial went a little bit further; in their discourse they included the words numerator and denominator. In the third column of Figure 5, they wrote: “before, we just eliminated the like terms from N/D [numerator over denominator] and now we know that if the numerator doesn’t have like terms then the expression cannot be simplified.” Compared to their written discourse shown in Figure 3, they had now begun to talk explicitly about the numerator and denominator and to speak about “like terms” instead of their very loose, and poorly understood, formulation involving “common factors.”

Expresión para la cual tu resultado con papel y lápiz no coincide con el dado por la calculadora en IIa)	Explica si es posible o no simplificar tales expresiones. Rehaz los cálculos, si es el caso.	En qué forma es diferente tu simplificación (segunda columna, si es el caso) respecto del procedimiento utilizado en IIa)
$\frac{2x+3y}{x}$ =	no se puede simplificar 2x + 3y	x -2x 2 0 3y que antes solo eliminabamos los	porque antes solo eliminabamos los terminos comunes del $\frac{N}{D}$ y ahora sabemos que si el numerador no tiene terminos comunes no se puede simplificar la expresion.

Figure 5: Explanation as to why the given expression cannot be simplified

However, for the expressions of the form "binomial over binomial" (see the last three expressions of Figure 4), their explanations were (at this moment of the activity) still evolving. The next verbatim extract illustrates this.

Researcher: I heard that you said that in this case it is possible to cancel out elements of the expression [*Referring to the last expression of Figure 4; immediately after they finished the long polynomial division*].
Student A: Yes.
Student B: Because there is a monomial in the bottom …
Student A: It is a binomial, isn't it?...
Researcher: So, why in the previous one [*Third expression of Figure 4*] is it that, that technique doesn't work?
Student A: Because there are not the same terms above and below [*Referring to the numerator and denominator*]
Researcher: And in the last [*Expression*] they are?
Student A: [*Nods his head in agreement*]
Researcher: Which ones are those terms you are referring to?
Student A: *3* plus *y* divided by *3* plus *y*.
Researcher: So, there [*Referring to the last expression for the Figure 4*] you identify that both techniques work, dividing or cancelling?
Student A: Yes, but here as well [*Signalling the second expression of Figure 4, and he tries to factor the expression*]… For which one you asked?…
Researcher: For the third one [*Referring to the third expression of Figure 4*]
Student A: Let's see… [*And he factors the expression, see Figure 4*]… Yes, you need to change the form [*of the expression*]
Student B: You factored the expression

After this, for expressions of the form "binomial over binomial" they explained their techniques in terms of factoring the expressions, even if for some cases there were still some inconsistencies in their explanations—that is, until they used the CAS for another case (see Figure 6).

Introduce en la calculadora	Resultado dado por la calculadora	Describe cómo harías para obtener el mismo resultado que el dado por la calculadora
$\frac{a+a\,x}{x}$	$\frac{a(x+1)}{x}$	Factorizando la expresion

Figure 6: CAS work

Once they used the CAS for simplifying the expression shown in Figure 6 and the CAS gave the result in factored form, this decisively changed their point of view regarding the technique for simplifying rational expressions. From then on, their explanations included the idea of factoring (as seen in the third column of Figure 6).

Conclusions

In this paper we have shown that the CAS environment led students to think in terms of factoring when simplifying rational expressions—something that they had not previously considered in their initial techniques of "cancelling" or using the "long division algorithm for polynomials." This is in contrast to the findings from our earlier pilot study (Guzmán, Kieran, & Martínez, 2011) where students did not possess both initial simplifying techniques and where their CAS work did not lead to the emergence of the idea of factoring and its role in simplifying rational expressions. While both studies provided evidence for the power of CAS to stimulate theoretical reflection, the findings of this study suggest that if students have more initial paper-and-pencil techniques (even if not completely understood), the CAS work can spur them to deeper theoretical reflections than for students who have fewer techniques.

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