FOSTERING STRATEGIC COMPETENCE FOR TEACHERS THROUGH MODELING RATIONAL NUMBERS PROBLEM TASKS

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The purpose of the study was to examine how teachers enhance their knowledge of rational numbers focused on modeling problem tasks using multiple representations. The professional development summer institute and the follow-up Lesson Study (Lewis, 2002) throughout the academic year focused on engaging teachers in rational numbers and proportional reasoning problem solving tasks, exploring pedagogical strategies, utilizing mathematics tools and technology, and promoting connections in the elementary and middle school curricula. This research report has two aims: (1) identify ways in which focusing on modeling rational numbers with multiple representations impacted teachers' understanding of rational numbers and proportional reasoning concepts; and (2) examine what strategic competence (NRC, 2001) looks like in teachers as they learn to model rational numbers concepts using multiple models.

Keywords: Modeling; Rational Numbers; Teacher Development; Professional Development

Theoretic Framework

Strategic competence has been defined as the "ability to formulate, represent, and solve mathematical problems" (NRC 2001, p. 116). The National Research Council define "mathematics proficiency" as having five strands that include strategic competence along with conceptual understanding, procedural fluency, adaptive reasoning and productive disposition. This study uses the term strategic competence as a competence we want to develop in teachers and expand the definition to include specific criteria in AMTE's standards for Pedagogical Knowledge for Teaching Mathematics, which include the ability to "construct and evaluate multiple representations of mathematical ideas or processes, establish correspondences between representations, understand the purpose and value of doing so; and use various instructional tools, models, technology, in ways that are mathematically and pedagogically grounded" (AMTE, 2010, p. 4). Modeling mathematics and developing representational fluency are key mathematics practices emphasized in the common core standards for math (CCSSI, 2010).

Research on rational numbers has also shown that representational fluency is critical in developing a conceptual understanding of the topic (Lamon, 2007; NRC, 2001). Representational fluency, the ability to use multiple representations and to translate among these models, has been shown to be critical in building students' mathematical understanding (Goldin & Shteingold, 2001; Lamon, 2001). The Lesh Translation Model highlights the importance of students' abilities to represent rational numbers in multiple ways, including manipulatives, real life situations, pictures, verbal symbols and written symbols (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Translations among the different representations assess whether a student conceptually understands a problem. Such abilities to be able to translate within and among multiple representations indicates an aspect of strategic competence. Some of the ways to demonstrate translation among representations in mathematics is to ask students to restate a problem in their own words, to draw a diagram to illustrate the problem, or to act it out. In teaching and learning, representations can play a dual role, as instructional tools and learning tools. As Lamon (2001) states, representations can be "both presentational models (used by adults in instruction) and representational models (produced by

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students in learning), which can play significant roles in instruction and its outcomes" (p.146). Another way to think about representations is that they allow for construction of knowledge from "models *of* thinking to models *for* thinking" (Gravemeijer, 1999). The *Principles and Standards for School Mathematics* (NCTM, 2000) emphasizes that representations serve as tools for communicating, justifying, sense making and connecting ideas by stating, "Representations allow students to communicate mathematical approaches, arguments, and understanding to themselves and to others. They allow students to recognize connections among related concepts and to apply mathematics to realistic problems" (p. 67).

Research Questions

This study explored the following research questions:

1. How does focusing on modeling rational numbers with multiple representations impact teachers' understanding of rational numbers and proportional reasoning concepts?
2. How do teachers exhibit strategic competence, in terms of the ability to construct, use and evaluate multiple representations and models of mathematical ideas and establish correspondences between representations?

Methods

Sixteen elementary and middle grades teachers from grades 3–8 met for a one-week summer institute and continued as school-based Lesson Study teams during the academic year. A majority of the teachers (78%) taught in Title One schools that served underrepresented and underserved populations. The daily topics included reasoning up and down, direct and inverse thinking, unitizing, and, ratios and proportional thinking. For this research report, we focused our analysis in the summer content institute and the lesson study data sources to demonstrate the progression of development in teachers' strategic competence as they emerged from the critical incidents as reported in teachers' reflections and instructors' memos and field notes and artifacts.

Data Sources

The data sources included teacher reflections, posters of solution strategies, videotapes of the class sessions, instructors' memos and field notes.

Teacher daily reflections from content institute. Teachers reflected daily on the problem solving tasks and wrote about new strategies and representations that were shared in class by other teachers. The teacher reflections focused on the understanding, reactions, and feelings of the individual teachers. The purposes of the daily reflections were to elicit responses in teachers focused on rational number problems which asked teachers to explain their thoughts and solution strategies; identify any differences in their own understanding, approaches, and thinking which resulted from the day's activities; and, illuminate any modifications to their teaching content and approach which they intend to employ.

Artifacts from class sessions–Poster proofs and concept map posters. The data collected, teacher reflections, posters of solution strategies, videotapes of the class sessions, and field notes, was focused on the development of conceptual knowledge, not procedural skill. Each group discussed the problem, and recorded their thought processes on large poster paper. The poster proofs were used to explain their reasoning, their discussions, their mistakes, and their conclusions with the class. Others in the class could comment or ask questions. These poster proofs used to document teachers' progression of ideas.

Video class sessions and instructors' memos. The researchers collected instructors' memos each day to serve as a record of the professional development. Researchers also took photographs and video recorded daily sessions focused on teachers' sharing their representations, class discussions, and studying teachers' work and collaborative poster proofs.

Through the use of multiple data sources our goal was to capture, teachers' strategic competence, namely: (a) the connection between teachers' content knowledge and the use of representations;

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(b) teachers' use of mathematics models, tools and technology and pedagogical strategies; and
(c) teachers' rationale for choosing tools and representations to represent their thinking. The researchers included the faculty and knowledgeable others who recorded their observations in a consistent format that helped us analyze and identify evolving themes and misconceptions.

Data Analysis Procedure

Critical Incident Analysis (Tripp, 1992, 1994) was used to analyze key events that evoked teachers to reflect on their math knowledge for teaching rational numbers and their teaching practices. Tripp (1992, 1994) defines critical incidents, which emerge through the critical reflection process, in the following way: *"Incidents happen, but critical incidents are produced by the way we look at a situation: a critical incident is an interpretation of the significance of an event. To take something as a critical incident is a value judgment we make, and the basis of that judgment is the significance we attach to the meaning of the incident."* Tripp also describes critical educational events are catalysts for transformative development of both students and teachers. As researchers, we took inventory of critical incidents that occurred throughout the content institute and the Lesson Study and collected the data sources from those episodes to analyze them for their meaning, relate the incidents to a broader analysis to understand how those critical incidents developed teachers' strategic competence.

Findings

Critical Incident 1: Letting Go of Formulas and Modeling Division of Fractions: What's All These Partitive, Quotitive Models?

A challenge that teachers encounter in their curriculum is having to model division of fraction. This requires understanding of the partitive and quotitive model of division. It was evident in many of our teachers that they had learned math rules without conceptual understanding and were challenged to reason about the mathematics they were teaching. To understand modeling division of fractions it is necessary to appreciate the different meanings such as measurement division, sharing, finding a whole given a part, and missing factors etc. Two different conceptual models that often evolve in modeling fractions include a fair-share (partitive) or measurement (quotitive) model. In the fair-share partitive model, the goal is to share out the same number of object to a fixed number of groups. On the other hand in a measurement quotitive model, a measurement unit is chosen and is repeated as many times to yield the quantity being measured. While the former leads to an invert and multiply algorithm, the latter leads to a common denominator algorithm. One of the instructors focused her module on this notion of division of fractions and in helping teachers model story structures that represented partitive and quotitive models. She writes in her instructor's memo:

> *I did see discussions between models of division that showed that participants did not have two equally robust models of division that they could use in their models. There was a debate between two participants that suggested that one participant had a model of division that was partitive, but the table-mate was showing a quotitive model. I hope that through modeling division problems tomorrow they will have the opportunity to figure out each division model from a set of problems. Having participants use manipulatives to model quotitive and partitive expressions challenges their views of division. (Excerpt-instructor's memo Day 3)*

> *The next thing I learned today is that having participants use manipulatives to model quotitive and partitive expressions challenges their views of division. For example, students who were comfortable with modeling $\frac{3}{4} \div \frac{1}{4}$ were stumped by $\frac{4}{5} \div 2$. However, the reverse was also true: participants comfortable with $\frac{4}{5} \div 2$, could not figure out a way to model $\frac{3}{4} \div \frac{1}{4}$. It is fascinating to me that this occurred at most tables, and I think it is something that could be followed up on. From a teaching point of view, setting up the confusion over division models and then resolving them by naming the modeling process made it much easier to teach the idea of partitive and quotitive. The participants knew that there was something "fishy" going on, but couldn't name it, and therefore couldn't work*

with their models. I know that many want to skip teaching these ideas explicitly, but I think it is an essential understanding in learning to model rational number operations, and more importantly, learning to teach students modeling! (Excerpt-instructor's memo Day 4)

These excerpts from the instructor's memo were revealing of how this class activity elicited a relearning experience for teachers. One teacher commented in her reflection, "I am grappling with the process of modeling the process of dividing by a fraction. Since I learned to multiply by the reciprocal over forty years ago, and it has always worked that way, I have never questioned that process really works. I still don't fully get it. But I will continue to examine the model until I "get it!" Other teachers also echoed their awareness of being too reliant on learned procedures and how they needed to let go of the formulas to relearn the conceptual models of operation with rational numbers. "I need to constantly use the manipulatives or I revert back to my happy place with algorithms."

Critical Incident 2: Developing Conceptual Maps and Poster Proofs with Multiple Models

The conceptual posters were used to document teachers' progression of ideas. In addition, posters showed how people in a group approached problem solutions in a variety of ways. The reflections gave insight into how the individual teachers were feeling about the sessions but they also documented how they were adding new models on to their conceptual maps for rational numbers. In class sessions, groups were required to strategize solutions by at least three of the five possible representations. The teams would affix their poster proofs to the wall, but, before verbal explanations from the teams, the class would do a "gallery walk," a walk around the room stopping to look at and to analyze each poster then they would take time considering different representations. Several themes were present in the majority of the reflections about the poster proofs. These were: the importance of clarity in the models, seeing the connection between the various models, the advantage of building multiple models, the benefit of collaboration, and recognizing that there are multiple valid approaches to problem solving, which leads to viewing student work with new eyes. Several teachers reported "Aha" moments concerning ideas about rational numbers, which they had formerly accepted but now actually understood, giving them a feeling of liberation. A teacher wrote, "I wish more classroom teachers fostered an environment where students can struggle with problems and work together to solve problems. Struggling through and listening to strategies of others has really opened up my thinking." As the teachers' conceptual knowledge deepened, the teachers began to question their own knowledge and assumptions. Classroom discussions of problems and sharing solution strategies was seen as a valuable approach both to clarify problems as well as to develop their conceptual thinking.

The teachers rediscovered the use of a ratio table to solve a problem called the Robot and Cars problem. Teachers reported that the reasoning up and down strategy helped them to break problems into chunks and build on those chunks. One teacher wrote that she would use reasoning up and down to help her students focus on what they already know and then guide them in building on that knowledge. Several teachers remarked on the importance of labeling processes so that students have a clear picture of how the concepts tie together; this leads to the development of conceptual understanding and the internalization of concepts and processes for the students. The teachers recognized the crucial importance of thinking about the question before crunching numbers. Additionally, as can be seen in the posters, the teachers gained an appreciation for the validity of multiple approaches to problem solution (see Figure 1).

One of the teacher's reflection commented on how the poster proofs allowed for colleagues to share different models of proportional reasoning. *"Even though people have different approaches on problem solving. Not one person thinks alike. The robot/hrs/cars problem had multiple ways to get the answer. Some were very basic and others more complex."*

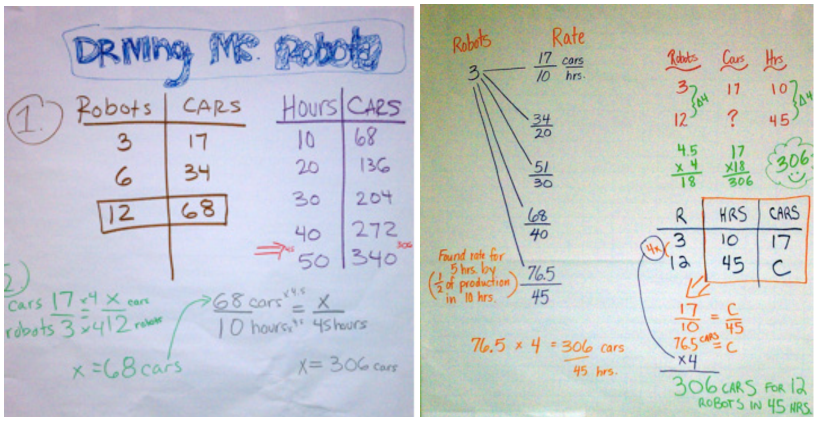

Figure 1: Using a ratio table and counting vs. finding multipliers

Mistakes and confusion allowed the teachers to use mathematical reasoning and arguments to do side-by-side comparisons of solutions, or just talk through comparisons of solutions to find where they did not match up. Then, the teachers would strategize to determine not only how to proceed but also to determine why one method did not work. For example, "1 robot can make 1 car in 1 hour" does not mean "2 robots can make 2 cars in 2 hours." Teachers discussed why a simple "multiply through" technique did not work. Teachers benefited from these discussions in several distinct ways. First, they began to see that real problems involving rational numbers are not simply plug-and-play exercises; they are multi-layered challenges, which require analysis, sound reasoning, and understanding of the relationships among quantities. Second, they recognized the profound importance of conceptual understanding as a baseline for strategizing approaches to problem solving. And, third, they gained an acute appreciation for the frustration of their students who apply incorrect procedures and cannot understand why their answers are incorrect. Several teachers mirrored that idea in their writings. Lastly, another teacher reflected, *"I am also starting to think differently about analyzing student work. When problems have the opportunity of yielding a variety of correct answers, it is important to consider what the student is doing and what math they can do and understand."*

Critical Incident 3: Using Lesson Study to Observe How Students Modeled a Problem

One critical class episode during the summer institute surrounded a problem called the *Mango Problem*. The problem is as follows: *One night, the King went down into the Royal kitchen, where he found a bowl full of mangoes. Being hungry, he took 1/6 of the mangoes. Later that same night, the queen was hungry, found the mangoes and took 1/5 of what the King had left. Still later, the first Prince awoke, went to the kitchen, and ate ¼ of the remaining mangoes. Even later, his sister, the Princess, ate 1/3 of what was then left. Finally, the youngest Prince woke up hungry and ate ½ of what was left, leaving only 4 mangoes for the kitchen staff. How many mangoes were originally in the bowl?* Teachers initially had difficulty approaching this problem because they were fixated on the whole being one mango or figuring out a formula. The researcher noted how the instructor reminded the teachers to "letting go" of rules and figure out ways to approach problems through modeling without getting fixated on the numbers. Video analysis revealed a group of teachers, Sunny, Jane and Al act out their solution. As they acted out the scenario, they asked questions like, "Is a mango the whole or are 4 mangoes the whole?" "What role are the fractions playing?" They started to wrestle with the idea of their previous math task called the Candy Bar and Circle Problem, which focused on the varying definitions of the "whole" and they had to negotiate and determine different meaning of fractions of that whole. Some teachers were observed having obstacles because they started with one mango as the whole; but, halfway through started to think as 4 mangos as the whole. This indicated a misconception that the teachers seemed to have about part-whole vs part-part interpretation. In addition, we observed teachers solving problems by working backwards using the manipulatives. The idea of unitizing that involves mentally constructing quantities in different chunks appeared to be somewhat problematic even for teachers. Although this group seems to be quick to catch on, they seem to be having problems truly grasping the concepts and applications of unitizing.

Because we noted this episode to be a critical incident, we were interested to see how this group of teachers who planned a Lesson Study (Lewis, 2002) with the *Mango Problem* would elicit models from their students. For the *Lesson Study Reflections,* we asked teachers to reflect on the process of developing and refining a research lesson, creating assessments items, and analyzing students' learning. The formal reflection assignment included teachers' evaluation of instructional strategies that promoted rational numbers and proportional reasoning through modeling, teachers' analysis of student thinking and what was learned from the process of collaboratively planning, teaching, observing and debriefing with colleagues. One of the Lesson Study teachers taught the mango problem to her 5th grade students and commented on the multiple models and representations that were used in her class.

> *Students approached the task in numerous ways. Some students tried to employ algorithmic approaches base on their current knowledge. This strategy often highlighted misconceptions they were having in regard to the relationships of fractions. Students would add all the numerators and then add on the number of mangos that remained. Others drew pictures or a model of 6/6 and took 1/6 away, but got stuck with where to go next. Others used the unifix cubes and represent this model the same way and were not sure how to proceed either. Still others quickly drew a model of 6/6 and identified the last box as having three mangos in it. They saw at that point that because fractional parts are of equal size all the boxes would have three mangoes in them. From there, they eliminated 1/6, 1/5,1/4, etc. recognizing that each time they took away one-sixth their whole changed. Drawing seemed to be the strategy that worked the best.*

Her reflection continued with an analysis of her students' work and how she asked her students to use their models of understanding the problem to justify their answers. This teacher reflected upon this Lesson Study and reported that the planning of the *Mango Lesson* helped bring deeper understanding of the importance of unitizing or the changing of the unit as one proceeds through a task. In addition, it solidified the meaning of fractional parts being of equal size. The planning session, also, brought to the forefront for her the multiple approaches that could be utilized by students to solve the task. Developing pictorial representations and then discussing the processing behind each solution with a collegial group allowed her to see thinking that was different from hers and yet valid. They looked at the process of working backwards and the relationship of parts to the whole. Collaboratively discussing misconceptions with her lesson study group also aided her in developing open-ended guiding questions to assist students in navigating through the task if and when they get stuck while modeling the task.

Discussion and Conclusion

Our study operationalized the notion of teachers' strategic competence using the NRC's (2001) description "as the ability to formulate, represent, and solve mathematical problems" and AMTE's standard (2010), "as the ability to construct and evaluate multiple representations of mathematical ideas or processes, establish correspondences between representations, and understand the purpose and value of doing so; and use various instructional tools, models, technology, judiciously, in ways that are mathematically and pedagogically grounded".

In our analysis we observed that teachers needed multiple opportunities to construct and evaluate multiple representations of mathematics ideas. In the critical incidence described above, teachers recognized that certain models afforded different opportunities for mathematizing. For example, the ratio table allowed teachers to bring out the ideas of reasoning up and down and highlight the multiplicative structures in proportional reasoning. In addition, the notion of "establishing correspondences between representations" came up a lot as an important theme when making connections between tabular, numeric and graphical approaches to representing a problem. *"Because I am so comfortable with mental math and using numbers, I find it arduous to think in terms of manipulatives and pictures. However, I can see the*

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value of hands-on manipulatives for my math students. Today I used a ratio table and Kathy showed me how to "pull apart" a ratio so that I could manipulate it more easily."

During the lesson study, the planning and debriefing phases revealed teachers pedagogical dilemmas with the "use various instructional tools, models, technology, judiciously, in ways that are mathematically and pedagogically grounded." For example, teachers who presented the mango problem wrestled with the pedagogical dilemmas of determining which manipulatives should be available for students and what model of fractions would be important in the lesson.

Most importantly, we gathered from their multiple reflective entries, teachers' sense of "understanding the purpose and value of doing so (representing and connecting representations). Teachers reflected on how the opportunity to struggle with problems in order to develop deep understanding of rational numbers. While many teachers expressed frustration with the homework problems as well as the in-class problems, they also recognized that their frustration led them to think about rational numbers in ways which they had not employed previously. This led to deeper understanding. Several teachers reported that they now "get" rational numbers and are gaining appreciation for the connections between concepts; they attribute this to the experiences of struggling through the investigative problems without the crutch of plug-and-play procedures. Teachers questioned each other's thinking and would not allow unsubstantiated assumptions. The focus was on mathematical reasoning, not the answer. We repeatedly heard teachers asking each other, "please explain that again, I don't understand where you are going with this" or "why would that be reasonable way to solve this?" Knowing that numerous approaches to problem solution were both possible and valid freed the teachers to concentrate on the soundness of their approaches, resulting in the teachers being able to develop more profound understanding. Participants valued the learning process and the opportunity to collaborate with other mathematics educators in translating their learning into practice. This study contributes to the growing body of knowledge on documenting how professional development serves as a catalyst for change in teachers as they reflect on developing their strategic competence for teaching and modeling rational numbers concepts in elementary and middle grades.

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References

Association of Mathematics Teacher Educators. (2010). *Standards for elementary mathematics specialists: A reference for teacher credentialing and degree programs*. San Diego, CA: Author.

Common Core State Standards Initiative (CCSSI). (2010). *Common Core State Standards for Mathematics*. Retrieved from http://www.corestandards.org.

Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representations in school mathematics* (pp.1–23). Reston, VA: National Council of Teachers of Mathematics.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, I*(2), 155–177.

Lamon, S. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. Curcio (Eds.), *The roles of representation in school mathematics. 2001 Yearbook* (pp. 146–165). Reston, VA: National Council of Teachers of Mathematics.

Lamon, S. (2007). Rational numbers and proportional reasoning. In F. K. Lester, Jr. (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 1169–1207). Reston, VA: NCTM.

Lesh, R., Cramer, K., Doerr, H., Post, T., & Zawojewski, J. (2003). Using a translation model for curriculum development and classroom instruction. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism. Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lewis, C. (2002). *Lesson Study: A handbook of teacher-led instructional change*. Research for Better Schools.

National Council of Teachers of Mathematics. (2000). *Principles and standards of school mathematics*. Reston, VA: Author.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, & B. Findell (Eds.), Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Tripp, D. (1993). *Critical incidents in teaching: Developing professional judgement*. London: Routledge.

Tripp, D. (1994). Teachers' lives, critical incidents and professional practice. *Qualitative Studies in Education*, *7*(1), 65–76.