YOUNG CHILDREN'S UNDERSTANDING OF ANGLE IN A DYNAMIC GEOMETRY ENVIRONMENT

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This paper examines the effect of the use of dynamic geometry environments on children's thinking about angle. Using a driving angle model in Sketchpad, kindergarten children were able to develop an understanding of angle as "turn," that is, of angle as describing an amount of turn. Gestures and motion played an important role in their developing conceptions.

Keywords: Geometry and Geometrical and Spatial Thinking; Elementary School Education; Angle; Gestures

Introduction

The concept of angle is multifaceted and can pose challenges to learners, even into secondary school (Close, 1982, Mitchelmore & White, 1995). Despite these difficulties, children show sensitivity to the concept of angle from very early years (Spelke, Gilmore, & McCarthy, 2011). Angles are normally introduced to children quite late in formal school settings. For example, in British Columbia, they are introduced in grade 6 (12 years old), even though students are expected to describe, compare, and construct 2-D shapes, including triangles, squares, rectangles and circles in grade 2. The strong capacity of young children to attend to and identify angles in various physical contexts motivated us to see whether a more dynamic conception of angle—namely, angle-as-turn—might support their developing understanding at an earlier age.

We have been investigating other geometry-related concepts at this age too, using DGEs, including shape identification, symmetry and parallel lines (Sinclair, Moss, & Jones, 2010; Sinclair & Kaur, 2011). Previous research reports on the effectiveness of Turtle Geometry (Logo) for teaching the concept of angle (Clements, Battista, Sarama, & Swaminathan, 1996; Simmons & Cope, 1990). However, we believe that DGE might be helpful in thinking of angles as turns and rotation more effectively. In this paper, we report on an exploratory study conducted with a split class of kindergarten/grade 1 children (ages 5—6) working with angle using *The Geometer's Sketchpad*. We focus on the emergence of the concept of angle-as-turn and discuss the specific mediating role of the use of the software on this thinking.

Children's Understanding of Angle

In the research literature, the concept of angle is shown to have different perspectives, namely: angle as a geometric shape, union of two rays with a common end point (static); angle as movement; angle as rotation (dynamic); angle as measure; and, amount of turning (Close, 1982; Henderson & Taimina, 2005). Due to different prevalent definitions of the term angle, teachers often face difficulty in knowing what definition of angle to use (Close, 1982). Mitchelmore (and colleagues) and Clements (and colleagues) have done abundant research in the area of angle concept over the past twenty years. Much research has been conducted on the development of the concept of angles, focusing at the grades 3, 4 and higher levels. Mitchelmore and White (1995) suggest that angles occur in a wide variety of physical situations that are not easily correlated. Despite the excellent knowledge of all situations, specific features of each situation strongly hinder recognition of the common features required for defining the angle concept (Mitchelmore, 1998). Mitchelmore and White (1995) proposed that children initially acquire a body of disconnected angle knowledge situated in a large number of everyday experiences; they then group situations to form angle contexts such as turns and corners; and finally they form an abstract angle concept by recognizing similarities across several angle contexts.

Later works of Mitchelmore involved teaching experiments (White & Mitchelmore, 2003) in which they divided angle situations into three clusters—2 line angles (corners of room, intersecting roads, pairs of

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scissors), 1-line angles (doors, windshield wipers), and 0-line angles (the turning of a doorknob or a wheel). The situation is more problematic for students where the two arms (of angle) are not clearly visible. Research using *Logo* shows that students tend to visualize the turn of turtle as turn of their body but making these turns involves writing numerical commands (Clements, Battista, Sarama, & Swaminathan, 1996). The DGE does not involve the writing of the commands and can thus be used at an earlier age to develop more qualitative understanding of angle. We believed that the DGE approach would be helpful in developing the dynamic as well as static concept of angle.

Theoretical Perspective

In previous research, we have found Sfard's (2008) "commognition" approach is suitable for analysing the geometric learning of students interacting with DGEs (see Sinclair, Moss, & Jones, 2010; Sinclair & Kaur, 2011). For Sfard, thinking is a type of discursive activity. Sfard's approach is based on a participationist vision of learning, in which learning mathematics involves initiation into the well-defined discourse of the mathematical community. The mathematical discourse has four characteristic features: word use (vocabulary), visual mediators (the visual means with which the communication is mediated), routines (the *meta-discursive rules* that navigate the flow of communication) and narratives (any text that can be accepted as true such as axioms, definitions and theorems in mathematics). Learning geometry can thus be defined as the process through which a learner changes her ways of communicating through these four characteristic features. We have previously presented a developmental trajectory related to identifying shapes in terms of different levels of discourse and now we are trying to do the same thing with angles, but we will look first at how the different components of the discourse change as the students work within the DGE. We are particularly interested in investigating how the students might move between different word uses and to examine the informal language they use to talk about angles.

Similarly, given the importance of gestures in communication of abstract ideas (Cook & Goldin-Meadow, 2006), and their potential to communicate temporal conceptions of mathematics (Núñez, 2003; Sinclair & Gol Tabaghi, 2010), we chose to extend Sfard's approach to incorporate gestural forms of visual mediators. Given the fact that we are working with very young children who have had little exposure to a mathematical discourse around angle, we will be interested in seeing whether they make use of kind of "mismatch" gestures that Goldin-Meadow (2004) describes as indicating a readiness-to-learn. Kita (2000) focuses on the cognitive functions of gestures, which play an important role in communication. He points out that the production of a gesture helps speakers organize rich spatio-motoric information, where spatio-motoric thinking organizes information differently than analytic thinking (which is used for speech). We thus expect that children will use gestures to convey spatio-motoric information, even though they might not be able to convey the analytic thinking used in speech. Moreover, children's gesture might be non-redundant with their speech. Our goal in looking at the gestures will be to see how they communicate different ideas about angles; particularly the mobile ones associate with angle-as-turn. We are less interested in classifying the students' gestures in terms of McNeill's (1992) categories than in understanding the embodied, conceptual basis of the gestures.

Exploring the Concept of Angle

Participants and Tasks

We worked with kindergarten/grade1 children (aged 5—6) from a school in a rural low SES town in the northern part of British Columbia. There are 20 children with diverse ethnic backgrounds and with a wide range of academic abilities. We designed lessons related to angle along with the classroom teacher, who has a Master's degree in mathematics education and has been developing her practice of using DGEs for a couple of years. The teacher and students worked with angles in different ways using Sketchpad for six lessons in a whole class setting with an IWB (Interactive Whiteboard). Each lesson lasted approximately 30 minutes and was conducted in a group with the children seated on a carpet in front of a screen. Lessons were videotaped and transcribed.

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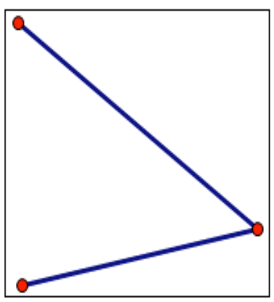
Dynamic Angle Sketches

In this study, we used two different sketches to explore the concept of angle with the children. We began with a simple angle diagram (Fig.1). In the sketch, dragging the vertex of an arm of the angle changes the angle. The purpose of using this sketch was to enable children to focus on the standard form of angle as a geometric shape and to build an understanding of angle through its behavioural properties. The research suggests that children have difficulty seeing a static angle as a turn. The situation is more problematic when the two arms of the angle are not clearly visible. The second sketch used is a "driving angle model," which shows both a static as well as dynamic sense of angle (Fig.2a, 2b). It includes a car that can move forward as well as turn around a point. The turning is controlled by a little dial (which has two arms and a centre). No numbers are used. There are four action buttons (Turn, Drive Forward, Erase Traces and Reset) that control the movement of the car. Students can regulate motion and turns to create different shapes like random paths, squares, rectangles, and so on.



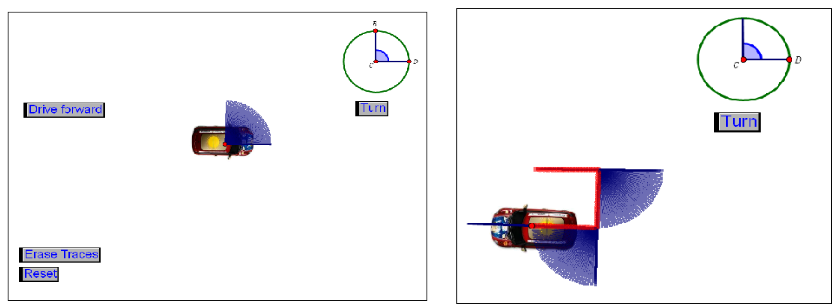


Figure 1: Angle as a Shape **Figures 2a and 2b: Driving Angle Model**

The traces offer a visible, geometric record of the amount of turn. The purpose of this sketch was to move to a more dynamic presentation of angle related to a real life context, where the focus of the children would be more on the continuous behaviour of the turning wheels and would enable them to see the process of turning along with the final product (position after a specific angle turn).

Classroom Discussion

The classroom teacher tried to support the discourse of angle-as-turn as she worked with the sketches. In what follows, we report on the children's work with the first sketch and then their developing sense of angle as they worked with the second one.

Introducing the angle as a geometric shape. The teacher began by showing the children the sketch in Figure 1 and asking them what they saw. Initially, the students attended to the features of the figure like points, circle etc. One student uttered the word "angle," although, when prompted, didn't elaborate. Two students, Colin and Jasmine, compared the sketch with triangle. Kristian was the first to impose motion on the diagram:

Teacher: Kristian, what do you see? (*Pointing towards figure1 on the screen*)
Kristian: I see the point is going back and that point is going up…*Inaudible*…

Kristian focussed here on the two arms of the angle, not mentioning explicitly the point at which they meet. The teacher asked about other similar examples and the children described what they saw in terms of concrete objects like a swing, slide, the letter "w," a house, the bottom of a hill, and a nose in their responses. Thus, initially their discourse was dominated by everyday language and they made little or no use of mathematical words—and their comparisons involved very loose visual resemblance. Following initial discussion, the teacher dragged the vertex of one arm of the angle.

Teacher: What happens if I do this? (*dragging one vertex of one arm of the angle*)
Will: It turns wider

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Teacher: It does turn wider. What about if I do this? (*dragging the vertex further*)
Morris: It widens even more now.
Teacher: It widens even more. What am I doing if I am moving this line? What kind of movement is that?
Jasmine, Will, Kristian: Square, it's a clock
Teacher: It's a clock movement?
Students: Yeah
Teacher: What am I doing to the line?
Students: Moving, clock.

When the teacher drags the vertex of one arm of the angle to make it wider, Will and Morris describe the change in terms of widening; Will also uses the word "turn" to describe the motion. Several students describe the movement of the arm of the angle as the movement of the clock. This is the first instantiation of angle-as-turn, which the students associated with a clock. Note that they do not explicitly use the word "angle," but they talk about the same diagram they saw before, now using an entirely new comparison, which becomes shared widely. The teacher finishes the lesson by talking about the space between the two arms and showing them how to mark an angle between two arms in *Sketchpad*, which can be done using the Marker tool.

Introducing driving angle model. In the subsequent lesson, the teacher presented the driving angle model to the students and asked them what they saw. Initially children focused more on features like color of lines, circle and points. The teacher prompted them to find the similarity between the two sketches (Figures 1 and 2a). Most of the students focused on describing the absence of the car in first sketch (Fig.1) and the presence of the circle in the second sketch (Fig. 2a). During comparison, only the second sketch was displayed on the IWB.

Will: And there is a blue one on this one
Teacher: There is a blue one on this one. That's the same?
Will: Yeah, Surrounded by a circle (*draws circle in the air to show the surroundedness. Repeats gestures a few times*)

Will recalled the angle diagram of first sketch and noticed its presence in both the sketches, but his description "blue one on this one" showed a dominance of everyday language. Despite the use of word "angle" by the teacher (while marking the angles) at the end of previous lesson, none of the students used the word in their comparison of the two sketches. During the classroom discussion, the children asserted that in order to drive the car, the wheels should be turned.

Teacher: Okay, I have another question for you guys. What does the car do?
Will: It drives on the ground. (*gesturing with both arms, turning both his arms together from left to right*)
Teacher: It does drive on the ground. So we're going to pretend our screen is the ground. How is it going to drive on the ground? What ways can it drive? Jasmine?
Jasmine: You have to turn the wheel. (*With both arms turning as if turning a steering wheel*)

Will and Jasmine both used gestures to describe the turning of the car. Many children in the class imitated their gestures. Jasmine used the gesture of turning the steering wheel as a visual mediator while associating the motion of car with the turning of wheels. The teacher then pressed the Turn button and the car turned by the same amount indicated by the angle dial, also leaving a trace of the turn behind.

Teacher: Okay…I am going to move something for you, here. Okay… If I tell you that this is kind of like the car's steering wheel, what do you think that car is going to turn like now? What do you think is going to happen? Think about it. (*Making the turning angle smaller in the size*), What is it going to do now?
Students: Turn, turn
Teacher: Turn…is it going to turn the same?

Students: Yeah.

The teacher told the students that the angle dial was like the car's steering wheel and she changed the size of the angle in the dial. Then she prompted the students to predict the outcome on pressing the Turn button; several students predicted that car would turn although they could not predict the amount of turn. This shows that students did not associate the turn of the car with the angle in the dial: they had no routine for assessing the magnitude of the turn. The teacher invited the children to compare the size of the angular turns. She changed the size of the angle and asked the children to decide whether this turn was smaller or larger than the previous amount of turn.

Teacher: The car is stopped…but if you look, that's like right here. The car is turned … turnn ... and stops here. And then I moved this steering wheel up here with the angle (Fig. 3a) and we press turn again and it only went from here to here (Fig.3b). Are these the same?

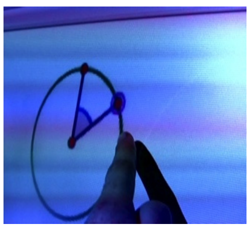


Figure 3a

Figure 3b

Figure 3c: Morris's gestures

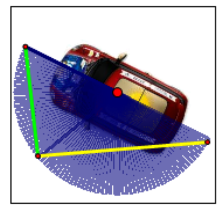


Figure 3d

Students: No, no, no…that one is too bigger (*pointing towards the screen*)
Teacher: Which one is bigger?
Morris: They don't match. There is no match. One is bigger (*Lying on the carpet, gesturing with his fingers as if he is comparing the sizes (Fig. 3c). And then shaking his head no*).

Morris compared the two different amounts of angle turns by comparing the size of the traces. Morris linked the difference in the size of two fingers (Fig. 3c and 3d) with the difference in two amounts of turn (sizes of traces). He used the gestures with two index fingers to explain his reasoning. These are new visual mediators that are used to express the idea of angle. They also point to a new routine for comparing "turn." His gesture here is non-redundant (see Kita, 2000) with his speech since he expresses meanings for angle/turn that are not evident in the speech. Indeed, the children use very few verbal expressions in these lessons. We note that the initial use of concrete visual mediators in first lesson was replaced by the use of embodied mediators in the form of gestures.

Understanding of benchmark angles. The teacher introduced some benchmark angles in the next lesson by using the first sketch with words like "right turn." She showed the 90° angle in the sketch (Fig.4a). The children compared the right angle with the side of a house and corner of a box. The teacher showed a whole movement of the one arms of angle from 0° to 180° by dragging the vertex of one arm (Fig. 4b).

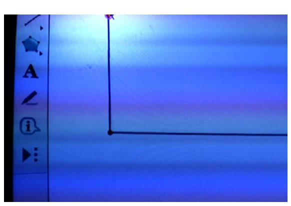


Figure 4a: Sketch showing 90^0

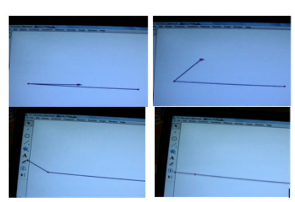


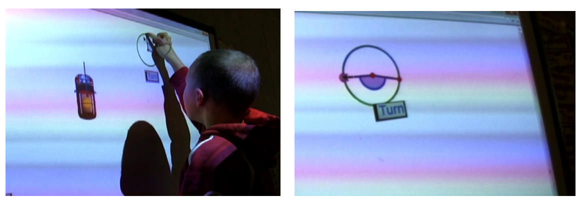
Figure 4b: Movement of one arm from 0^0 to 180^0

The children compared 180° to a line and a road. Morris made a gesture with his hand moving along a straight path, while describing a car moving along a straight road (Figure 5a)—thus picking up the context of the previous day's work. The teacher asked the students how they could make the car turn by 180^0.

Teacher: How can you make that car go that far…go in a straight line…Remember we did the 180 that was a straight line on the board. You said that it looked like a road. What can you do to the steering wheel?
Students: Press it
Teacher: No
Will: Move it
Teacher: How can you move it, Will? Show us. Could you show us that up there, Will? Give it a try. Can you show how to make it go 180? Show us how to go halfway around? (*Will comes on the board and try to move the vertex (point) on angle dial to adjust angle, but fails to move the point* (Fig.5b)). You have to hold it. Tell me when to stop? (*Will hold the vertex while dragging and adjusts angle to approx.* 180^0 (Fig.5c))

None of the children, other than Will, were able to recognize that they needed to change the angle in the dial in order to make the car to turn by 180°. Will showed an understanding of the association of the turn of car with the angle in the dial. This shift in the Will's understanding might be the result of seeing the teacher repeatedly changing the angle in the dial during the lesson and of seeing her systematic dragging shown in Fig. 4b.

At the teachers' request, Jasmine made the car turn halfway by pressing the Turn button (Fig. 5d). When the teacher asked about the direction of the car after pressing the Turn button again, both Chloe and Jasmine predicted that the car would complete the full turn.

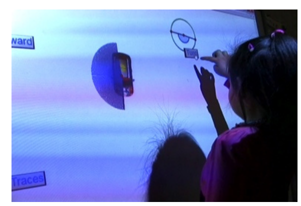


Figure 5a **Figure 5b & 5c: Will adjusting dial to 180^0** **Figure 5d**

Teacher: Why do you think it is going to go all the way around?
Chloe: Because when you press it, it will go…and stop
Teacher: Okay, can you think why it is going to do that?
Chloe: Because it goes that way and stops right there. (*Pointing towards the screen and reflecting, as if imagining the turn* (Fig. 6a))


Figure 6a: Chloe while responding **Figures 6b & 6c: Jasmine gesturing half and full turn**

From Chloe's explanation (including her gestures), it seems that she imagines the car turning and explains her reasoning in terms of the stopping point of the wheel.

Teacher: It is going to close. Why do you think it is going to close?
Jasmine: Because it is like this [*gestures shown in Fig. (6b, 6c)]*
Teacher: Yeah, because it is half way now…oh…that's interesting

While Chloe and Jasmine both visualised the final position of the car successfully, Chloe used words to describe the turn of the car, whereas Jasmine used the gestures to explain the reasoning in terms of half turning and full turning position. Both seemed to understand that two half turns result in a full turn. Once again, their understandings were communicated by the non-redundant gestures, in which arms are used as sides of angle, which provided new visual mediators for their reasoning.

Discussion and Conclusion

At the beginning of the episodes, the children had virtually no mathematical discourse around angle. By working on situations involving turn, which we thought would provide a strong embodied connection for the children, the goal of these lessons was to see whether students could develop a more sophisticated discourse around angle-as-turn—and not necessarily involving any numerical quantification of angle.

The dragging of the one arm of the angle focused the children's attention on the turning behavior of the segment in the diagram. This seemed to act as a visual mediator to which the children associated the movement of a clock. The clock movement metaphor enabled the children to see the angle diagram as turning of arms, and hence, initiated the discourse angle-as-turn. When the children were asked to work with the Drive model, they initially used informal ways to describe the turning of the car by gesturing the turning of a steering wheel. This steering wheel metaphor differs in important ways from the clock metaphor. While the latter focuses on the changing position of the arms (and thus on the changing "size" of the angle between the arms), the former focuses on the angle as a movement from one arm to another (and thus, on the "size" of a given angle).

Only one student was able to associate the turn of car with the angle dial explicitly. Will's recognition of the need to adjust of angle dial to 180^0, in order to move the car halfway, shows that the difficulty in visualizing the 0-arm angle—as reported by White and Mitchelmore (2003)—could be eased by the trace feature of *Sketchpad*, enabling students to see the process of turning as well as the product. Some children were successful in comparing two different amounts of turns. Also, some children were able to reason about half and full turns—in fact, they could explain that a full turn would require the repetition of two half turns. They could not talk about angle and turns using mathematical vocabulary.

At the beginning, the students described the angle diagram in terms of its parts, using concrete, everyday language; they eventually came to associate the notion of "angle" and "turn" to both a clock and a steering wheel—both of which capture something about the idea of angle-as-turn. Several students also used gestures to express the notion of angle, as can be seen in Figures 3 and 6. Finally, the students could also talk about angle in terms of various sizes (especially, the half angle). We argue that the gestures used by the students became part of their visual mediators for the concept of "angle." These gestures were at least in part evoked by the diagrams in *Sketchpad*. For example, the two index fingers shown in Figure 3 arise from the angle dial used by the teacher. The children's use of "turning" gestures in their responses to the car sketch shows that they created gestures as mediators for the purpose of communication. Also, Jasmine's gesture (Figure 6) relating half turn to full turn is a clear example of the effect to dynamic visualization offered by Sketchpad. The traces of half turn in *Sketchpad* acted as a visual mediator and then Jasmine used her gestures of half turn (straight opened arms) to make a full turn (covering the whole movement of arms to close hands) as a means of communication. This interplay between gesture and diagram resonates with Châtelet's (1993) theory of mathematical inventiveness, which de Freitas and Sinclair (2012) discuss in the context of mathematics education; in this case, the dynamism of the diagram seems to evoke quite directly the moving of the children's fingers and hands.

At this early stage, we have focused on word use and visual mediators. There was little sense of any routines used for identifying angles or evaluating their size. Further, given the emergent sense of angle and turn, there weren't any endorsed narratives. However, the frequent use of non-redundant gestures by Morris, Will, Chloe and Jasmine during the lessons indicates, as well as the relatively infrequent use of verbal expression, suggests that the children had a certain "readiness to learn" (Goldin-Meadow, 2004) about angle-as-turn. The next phase of our research, for which we are now in the process of gathering data, involves exploring the way in which the children's emergent sense of angle can be developed into routines for identifying angles and talking about their size. We have some initial evidence that the children are able to identify two angles as being the same even when their arms have different lengths because they use a routine of comparing turn. This would be a very important result given the existing research that shows that children often confuse the size of an angle with the length of its arms (Stavy & Tirosh, 2000).

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