MATHEMATICAL KNOWLEDGE FOR TEACHING HIGH SCHOOL GEOMETRY

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This paper documents efforts to develop an instrument to measure mathematical knowledge for teaching high school geometry (MKT-G). We report on the process of developing and piloting questions that purported to measure various domains of MKT-G. Scores on the final set of items had no statistical relationship with total years of experience teaching, but all domain scores were found to have statistically significant correlations with years of experience teaching high school geometry. We use this result to propose ways of conceptualizing how instruction-specific considerations might matter in the design of MKT items.

Keywords: Mathematical Knowledge for Teaching; Geometry; High School Education; Assessment

Overview

In his description of paradigms for research on teaching, Shulman (1986a) had called for a focus on teacher knowledge. With particular reference to mathematics, Ball, Lubienski, and Mewborn (2001) responded to Shulman's call by, on the one hand, reviewing research that showed that traditional measures of teachers' content knowledge (e.g., degrees obtained or mathematics courses taken) had not shown to make a difference on students' learning and, on the other hand, arguing that the kind of teacher knowledge needed to focus on was a particular kind of mathematical knowledge, mathematical knowledge for teaching (MKT). This MKT is knowledge of mathematics used in doing the work of teaching and it includes but also goes beyond the pedagogical content knowledge that Shulman (1986b) himself had proposed. The theoretical and empirical work on Ball's brand of MKT that followed such proposal has been vast, showing among other things that the possession of MKT can be measured, that MKT is held differently by teachers and non-teachers, that MKT is held differently by teachers of higher grade level experience than those of lower grade level experience, that it makes a difference in students' learning, and that scores on MKT correlate with scores on an observation measure of good teaching (Hill, Schilling, & Ball, 2004; Hill, Rowan, & Ball, 2005; Hill et al., 2008). The work on constructing measures of MKT has been concentrated mostly on the mathematical knowledge of elementary and middle school teachers (Hill & Ball, 2004; Hill, 2007); a more recent effort has developed MKT items in algebra (Mark Thames, personal communication, 6/15/11). The purpose of this paper is to report on a parallel effort to develop an instrument that measures mathematical knowledge for teaching high school geometry. Our effort has attempted to follow the theoretical conceptualization of MKT and item development procedures of Ball and Hill's group. The paper provides pilot data that compares high school teachers with and without experience teaching geometry in terms of their possession of mathematical knowledge for teaching geometry, and it uses these results to raise some questions about the content specificity of the notion of mathematical knowledge for teaching.

A crucial element in our development of items to measure the mathematical knowledge for teaching high school geometry has been Ball, Thames, and Phelps (2008) conceptualization of the different domains of mathematical knowledge for teaching. According to Ball et al. (2008), the mathematical knowledge used in teaching can be conceptualized as the aggregation of knowledge from six domains. These domains include Common Content Knowledge (CCK), which is the mathematical knowledge also used in settings other than teaching, including for example knowledge of canonical methods for solving the problems teachers assign to students. The domains also include knowledge that is specific to the work of teaching. Thus Specialized Content Knowledge (SCK) is knowledge of mathematics used particularly in doing the tasks of teaching, such as, for example, the knowledge a teacher needs to use in writing the problems they will assign to students or figuring out "whether a nonstandard approach would work in general" (Ball, et al, 2008, p. 400). A third domain, KCT, or Knowledge of Content and Teaching is defined as a combination of knowledge of teaching and knowledge of mathematics and includes the knowledge needed

to decide on the best examples and representations to use for given instructional objectives. And KCS, or Knowledge of Content and Students, includes a blend of knowledge of mathematics and of students' thinking, such as the capacity to predict what students might find confusing or what kind of errors students might make when attacking a given problem. In our effort to construct measures of mathematical knowledge for teaching high school geometry, we developed items that purport to measure each of those four domains CCK, SCK, KCT, and KCS. Ball et al. (2008) also include Horizon Content Knowledge (HCK) and Knowledge of Content and Curriculum (KCC), but our work has not included those domains.

Ball and Hill's Learning Mathematics for Teaching project has developed items that measure the different domains of MKT and that has included, over time, attention to different content strands, including number and operation, patterns, functions and algebra, and geometry. These instruments have also included items that purport to measure mathematical knowledge for teaching middle school mathematics as well as for teaching elementary school mathematics. The extensive item development has yielded numbers of validated items that can be put together into forms that assess MKT for particular content strands. But there has not been, as of yet, a systematic development of items to measure MKT in different content strands or deliberate theoretical consideration about how content-strand differentiation might interface with the domains of MKT (Heather Hill, personal communication, 2/8/12). In particular, how would the specific practice of teaching particular mathematics courses be considered and featured in the process of designing measures of the mathematical knowledge for teaching those courses? In this paper we present our beginning attempts to conceptualize such instruction-specificity within the framework of MKT, by reporting on our development of an instrument to measure the mathematical knowledge for teaching high school geometry.

Our interest in MKT originated from our attempts to contribute to a theory of mathematics teaching that accounts for what teachers do in teaching in terms of a combination of, on the one hand, individual characteristics of practitioners and, on the other hand, practitioners' recognition of the norms of the instructional situations in which they participate and of the professional obligations they must respond to (Herbst & Chazan, 2011). While our earlier work focused completely on the conceptualization and empirical grounding of the latter, the present effort was part of a larger project in which we'd develop measures of the constructs that we had contributed (particularly *norms* and *obligations*) as well as measures of other constructs that would give us measures of individual resources. The conceptualization and disciplined approach to measuring MKT spearheaded by Ball and Hill (Ball et al, 2008; Hill and Ball, 2004) provided us with important guidance for the development of MKT measures. Hence, we developed multiple choice items following the definitions of the domains provided by Ball et al. (2008).

Development of MKT-Geometry

Our item development process covered a relatively wide range of topics from the high school geometry course. We consulted curriculum guidelines in various states and on that basis sought to develop items dealing with definitions, properties, and constructions of plane figures including triangles, quadrilaterals and circles, parallelism and perpendicularity, transformations, area and perimeter, three-dimensional figures, surface area and volume, and coordinate geometry. Those topics by themselves were good enough a guide to create items of Common Content Knowledge. But the definitions of the MKT domains, particularly the definition of Specialized Content Knowledge, calls for items that measure knowledge of mathematics used in the tasks of teaching. To draft these items we found it useful to create a list of tasks of teaching in which a teacher of geometry might be called to do mathematical work. The list included elements like designing a problem or task to pose to students, evaluating students' constructed responses, particularly student-created definitions, statements, explanations, and arguments, creating an answer key or a rubric for a test, and translating students' mathematical statements into conventional vocabulary. As we sought to draft these items, we noted that those tasks of teaching could call for different kinds of mathematical work depending on specifics of the work of teaching geometry. For example, the task of designing a problem would involve a teacher in different mathematical work if the designed problem was a proof problem versus a geometric calculation. While the former might involve the teacher in figuring out what the givens should be to make sure the desired proof could be done, the latter might

involve the teacher in posing and solving equations and checking that the solutions of those equations represented well the figures at hand. Thus while a list of generic tasks of teaching was useful to start the drafting of items, this list appeared to grow more sophisticated with attention to tasks that are specific of different instructional situations in geometry teaching (Herbst, 2010).

The tasks of teaching were also useful in drafting items that measured knowledge of content and teaching. To draft these items we used as a heuristic the notion that the item should identify a well-defined instructional goal and the possible answers should name mathematical items that, while correct in general, would be better or worse choices to meet the specified goal. For example, teachers often need to choose examples (and justifications) for the concepts (viz. statements) they teach. While different examples (viz. different justifications) might be mathematically correct, they might not all be pedagogically appropriate to meet particular instructional goals: One example may be better than others as a first or canonical example while another example may be better as an illustration of an extreme case; one argument may require less prior knowledge and thus be more appropriate when students don't know many of the properties of the figure at hand, while another argument may illustrate how all the properties of a figure interrelate.

Finally, to create items that measured knowledge of content and students, we were attentive to the definition provided by Ball et al. (2008) and sought to draft especially items that tested for knowledge of students' errors. As in the case of other domains, there were specifics of the high school geometry class that shaped the items we developed. Thus, while we did create items that probed for teachers' knowledge of students' misconceptions about geometric concepts (e.g., angle bisector), we also created items that probed for their knowledge of students' misconceptions about processes or practices that are specific to geometry—such as the notion that empirical evidence is sufficient proof or that definitions are exhaustive descriptions.

While proving a claim on the board about the figure below, Joe wrote "1 + 2 = 90." Ms. Staples ponders how to correct that statement. Of the following, what is the best alternative?
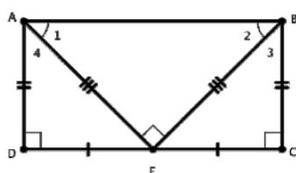
A	Do nothing. The statement is correct as is.
B	Add the degree symbol so that the statement reads $1 + 2 = 90°$.
C	Replace what Joe wrote; write instead that "$m\angle A + m\angle B = 90°$".
D	Replace what Joe wrote; write instead that "$m\angle EAB + m\angle EBA = 90°$".

After teaching her students the base angles theorem (which states that base angles of an isosceles triangle are congruent), Ms. Wellington is pondering what example she could give to illustrate how this theorem can help find new information. Which of the following would be the best example?

A	If you have a triangle that you know is isosceles and you know two of the sides are 8 and 10, then the third side will be 8 or 10.
B	If you have a triangle that you know the sides have length 8, 10, and 12 you know it does not have two angles that are congruent.
C	If you have an isosceles triangle and two of its angles are 30 you can calculate the third angle by subtracting 60 from 180.
D	If you have a kite and draw its minor diagonal, the kite is divided into two isosceles triangles and those have two congruent angles each.

Figure 1: Example of an SCK-Geometry Item (left) and KCT-Geometry Item (right)

Our research group drafted and revised an initial set of questions including 13 CCK, 20 SCK, 26 KCT, and 16 KCS questions; this drafting and revision process relied among other things on general guidance and comments on specific items by Deborah Ball, Hyman Bass, Laurie Sleep, and Mark Thames.[2] The questions drafted took the form of multiple-choice items, as well as multiple-response items within a single question (e.g., a single stem with 3–4 yes/no questions following). These items were submitted to a process of cognitive pretesting (Karabenick et al., 2007), by way of interviewing teachers and asking them to comment on what they thought each item was asking. Data from the cognitive interviews was also used to examine the content validity of the items, as well as improving such validity. Items were revised to improve interpretability and validity. A revised set of items was pilot tested with inservice secondary mathematics teachers from the same Midewestern state between July and October of 2011. Ten questions from each domain were uploaded into the LessonSketch online platform and completed by participants who took them either by coming in person to a computer lab (37 participants) or by responding to the items online from their homes or workplace (10 participants). For the purposes of this chapter, all data reported is pooled from both samples ($n = 47$). Participants were predominantly Caucasian (96.4%) and

female (56.4%). Participants varied in the amount of mathematics teaching experience ($M = 13.02$, $SD = 7.30$), mathematics content courses ($M = 10.78$, $SD = 4.46$), and mathematics pedagogy courses ($M = 3.04$, $SD = 2.54$). Additionally, 67% of participants had taught Geometry for 3 years or more. Participants completed other questionnaires including one in which they reported on their years of experience teaching secondary school mathematics and teaching high school geometry. Our goal was to use the pilot to select five questions from each domain, as well as additional public-release items.

Item analysis for the MKT-Geometry test was conducted separately for each domain (CCK, SCK, KCS, KCT). We also used the pilot data to select the public-release questions (see Figure 1). In examining the fit of items for each domain, we used biserial correlations (Crocker & Algina, 2006) to measure item discrimination or how well the items discriminated between higher scoring test-takers and lower scoring test-takers. Crocker and Algina (2006) note that in performing classical item analysis such as the one we present here should "…have 5 to 10 times as many subjects as items" (p. 322). Since we conducted item analysis per MKT domain, this suggests a sample of approximately 50 participants (5×10 items per domain).

The item analysis of all 10 CCK questions yielded an initial Cronbach's alpha coefficient of .54. We used low biserial correlations (below .30) as one indicator for possible item removal. This resulted in the removal of 3 questions and an acceptable level of internal reliability ($\alpha = .64$). The final set of seven questions had biserial correlations ranging from .30 to .48, suggesting sufficient item discrimination. Additionally, item difficulty, in the form of percentage of the sample selecting the 'correct' answer, ranged from 30% to 83%.

We applied the same process to the study of the 10 questions that purported to measure the SCK domain. Item analysis resulted in the removal of three questions. The internal reliability of the remaining questions was found to be sufficient ($\alpha = .68$), with item difficulties ranging from 19% to 96%. These results suggest both sufficient item discrimination and range of difficulty levels. Item analysis for the KCT domain led to removal of 3 items with a Cronbach's alpha of .57 with item difficulties ranging from 17% to 60%. Item analysis of the ten KCS items resulted in the removal of 3 items. Item difficulties ranged from 17% to 74% ($\alpha = .62$).

Table 1: Composite Scores and Descriptive Statistics

Domain	M	SD	N	α
CCK – Geometry	0.68	0.22	48	0.64
SCK – Geometry	0.64	0.19	48	0.68
Subject Matter Knowledge of Geometry (CCK & SCK)	0.66	0.18	48	0.74
KCT – Geometry	0.39	0.24	47	0.57
KCS – Geometry	0.44	0.25	47	0.62
Pedagogical Content Knowledge of Geometry (KCT & KCS)	0.41	0.21	47	0.66
MKT – Geometry	0.54	0.18	47	0.84

Table 2: Correlations Between MKT-G Domain Scores

	CCK	SCK	KCT	KCS
CCK	-			
SCK	.44**	-		
KCT	.41*	.59**	-	
KCS	.68**	.55**	.48**	-

*$p < .05$, **$p < .01$*

Items chosen through item analysis were used to compute scores for each domain (CCK, SCK, KCT, and KCS) shown in Table 1 above. Correlations between the domain scores are presented in Table 2, and suggest moderate to strong relationships between the different domains. These results show similar trends to those found by Hill et al. (2004) for CCK and KCS, which suggest that the different domains are, to a degree, interrelated. Thus, we interpret the findings from Table 2 similarly in that such relationships make sense, as it would be unusual for a teacher with higher KCS or KCT scores to have significantly lower CCK and SCK scores.

Following the notion proposed by Ball et al. (2008) that some of the MKT domains (notably KCS, KCT, and Knowledge of Content and Curriculum) operationalize the notion of Pedagogical Content Knowledge while the other MKT domains (CCK, SCK, and Horizon Content Knowledge) operationalize Subject Matter Knowledge, we created two additional scores: PCK-G which aggregates scores in KCT and KCS and SMK-G which aggregates scores in SCK and CCK. These are also summarized in Table 1.

Relationships Between MKT-G Scores and Teaching Experience

Our interest in MKT contributes to a larger project that investigates the influence that individual factors (such as mathematical knowledge for teaching) and socialization to the work demands of teaching a particular high school course (in this case, high school geometry, as indicated by teachers' recognition of instructional norms and professional obligations) have in the decisions that a teacher would make. A question we posed to the pilot data is what is the relationship between mathematical knowledge for teaching geometry and experience teaching the high school geometry course. Therefore, we correlated scores for each domain with teachers' years of experience teaching high school, but also with teachers' years teaching mathematics in general. These results are presented in Table 3.

Results indicated a statistically significant and positive relationship for each domain examined. These results show that the more years of experience a participant had teaching high school geometry, the higher their scores were for each domain. While that relationship was statistically significant for years of experience teaching geometry, such a relationship was not found to be statistically significant, or particularly meaningful in size for most measures, when looking at years of experience teaching mathematics in general. Therefore, these results suggest that while teaching experience may affect MKT-Geometry scores, it is the particular experience of teaching the geometry course. To the extent that mathematical knowledge for teaching is the knowledge of mathematics used in the work of teaching, the results lead to ask how the specifics of the instructional work a teacher does in a course matter in the mathematical knowledge for teaching the teacher has.

Table 3. Correlations Between Experience-Type and Score

	Years Teaching Geometry	Years Teaching Mathematics
CCK-G	.32*	.03
SCK-G	.31*	.11
SMK-G	.37**	.08
KCT-G	.36*	.27
KCS-G	.37*	.13
PCK-G	.42**	.23
MKT-G	.43**	.17

*p < .05, **p < .01*

Discussion and Conclusion

In our earlier and parallel work we have argued that the particular nature of the didactical contract (Brousseau, 1997; Herbst & Chazan, in press) for a course creates conditions of work that make the teaching of geometry different than the teaching of other mathematics courses, including algebra. The data

shown above seems to suggest that teachers of geometry have more mathematical knowledge for teaching geometry, while the difference does not seem to be accountable to general experience teaching secondary mathematics. While at one level one might not find that result surprising, the fact that three of the four domains of mathematical knowledge for teaching we tested for (SCK, KCT, and KCS) are defined as mathematical knowledge used in the work of teaching helps raise questions for future inquiry.

As we noted above, the current conceptualization of MKT has not addressed content differentiation within domains. A natural way of thinking about differentiation could be the topical content of the item—items drawing on knowledge from different branches of mathematics might aggregate into different scores. But that approach seems to apply well only to differentiation within the domain of Common Content Knowledge. To the extent that the other domains are defined in relation to the work of teaching, it is plausible that differentiation within each domain will require considerations of the specifics of the teaching involved. The results from this study suggest that the teaching of high school geometry may entail specific mathematical knowledge demands.

In particular, SCK is defined as the knowledge of mathematics used in doing the tasks of teaching. One could expect that some of those tasks will not be course specific: The task of creating a grading system, for example, involves a teacher in making a mathematical model that feeds from grades in individual assignments; but there is no reason for this mathematical work to be different for teachers of different high school courses. Other tasks of teaching, however, while amenable to generic statement (e.g., choosing the givens of a problem for students), may involve practitioners in different mathematical work depending on the specifics of the task (e.g., choosing the numbers for a word problem in algebra involves different mathematical work than constructing a geometric diagram to include in a geometry worksheet). Are those differences merely differences in mathematical strand (algebra vs. geometry) or do they also reflect differences in the instructional situations (Herbst, 2006) to which those tasks contribute?

We suggest that the management of instructional situations involves teachers in singular mathematical work. An instructional situation has been defined (Herbst, 2006) as a frame for exchanges between types of mathematical work that students will be doing and the knowledge claims that a teacher can make on their behalf based on their accomplishing that work. The teacher's management of instructional situations includes in particular the choosing of the various tasks that constitute that work, the observation of the proceeds (what students actually do), and the effecting of exchanges between such observed actions and the knowledge at stake (identifying at least for herself but possibly also publicly to the class how what students have done indicates their knowledge of what is at stake). While the definition of these tasks of teaching is general, the mathematical knowledge called forth in doing them would be different across different courses, as long as the specific exchanges were different.

A case in point that helps argue that instructional situations matter comes from one SCK question in our instrument. This was a multiple-response question with two items; the stem spoke of a teacher needing to choose algebraic expressions for the sides of an isosceles triangle where the students would be expected to find the lengths of the sides of the triangle after solving an equation. Each item provided expressions for the three sides and asked whether or not they were appropriate expressions. A quick examination of the responses to the item indicated that teachers with more or less years experience teaching geometry (≥ 3 years and < 3 years, respectively) did not respond much differently for the item where the equation could not be solved. However, the two types of teachers' responses did show differences for the item where the equation could be solved: the less experienced teachers tended to answer that the expressions were appropriate while experienced teachers that they were not. In fact, the numbers obtained after solving the equation would not work to represent the sides of a triangle in that the triangle equality would not hold for those numbers. We conjecture that the experienced teachers' familiarity with the instructional situation of "calculating a measure" (Herbst, 2010) mattered in their decision to check that the expressions would yield sides with positive lengths and that they would satisfy the triangle inequality. Our conjecture is not that the non experienced teachers did not know the triangle inequality, but that they did not know it mattered in this task of teaching, possibly because they only saw the problem as an exercise in algebra rather than also as an exercise in triangle properties. More generally, we conjecture that tasks of teaching that are subservient to instructional situations specific to a given course of studies might involve teachers in

mathematical work that teachers who are experienced in managing those situations would know better how to do. We suggest that considerations of the nature of the instructional situations in a course could lead to analogous differentiation within the domains of KCT and KCS as well.

Endnotes

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[2] Daniel Chazan, a co-PI of this project was also involved in design discussions. Individuals involved in the drafting of items included Michael Weiss, Wendy Aaron, Justin Dimmel, Ander Erickson, and Annick Rougee.

References

Ball, D. L., Lubienski, S. T., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433–456). New York: Macmillan.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Brousseau, G. (1997). *Theory of didactic situations*. Dordrecht, The Netherlands: Kluwer Academic.

Crocker, L., & Algina, J. (2006). *Introduction to classical and modern test theory*. Mason, OH: Wadsworth.

Herbst, P. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education, 37*, 313–347.

Herbst, P., with González, G., Hsu, H. Y., Chen, C., Weiss, M., & Hamlin, M. (2010). *Instructional situations and students' opportunities to reason in the high school geometry class*. Manuscript. Deep Blue at the University of Michigan, http://hdl.handle.net/2027.42/78372

Herbst, P., & Chazan, D. (2011). Research on practical rationality: Studying the justification of actions in mathematics teaching. *The Mathematics Enthusiast*, *8*(3), 405–462

Herbst, P., & Chazan, D. (in press). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM Mathematics Education*.

Hill, H. (2007). Mathematical knowledge of middle school teachers: Implications for the No Child Left Behind policy initiative. *Educational Evaluation and Policy Analysis*, *29*(2), 95–114.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*(5), 330–351.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and mathematical quality of instruction: An exploratory study. *Cognition & Instruction*, *26*(4), 430–511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371–406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal*, *105*(1), 11–30.

Shulman, L. S. (1986a). Paradigms and research programs in the study of teaching: A contemporary perspective. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed.) (pp. 3–36). New York: Macmillan.

Shulman, L. S. (1986b). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.