TWO FORMS OF REASONING ABOUT AMOUNTS OF CHANGE IN COVARYING QUANTITIES

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This paper addresses how secondary students might reason about amounts of change in covarying quantities. Two empirically based forms of covariational reasoning are distinguished. The first form—reasoning about quantities as varying simultaneously and independently—supports tandem comparison of amounts of change. The second form—coordination of change in one quantity with change in a related quantity—supports coordinated comparison of amounts of change. By expanding the mental actions of Carlson et al.'s (2002) covariation framework, these forms of reasoning provide finer grained distinctions in the "Quantitative Coordination" level of covariational reasoning. Distinctions made between these forms of reasoning might help to explain how students could begin from informal reasoning to transition to more formal reasoning about average and instantaneous rate of change.

Keywords: Algebra and Algebraic Thinking; Reasoning and Proof; High School Education

A student reasoning covariationally would be mentally "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p. 354). By conducting fine-grained investigations with secondary students, researchers have articulated the nature of relationships that students might make between covarying quantities (Johnson, 2012; Saldahna & Thompson, 1998). These articulations provide landmarks within a continuum of reasoning about covarying quantities.

This paper draws on two empirically based forms of secondary students' reasoning about amounts of change in covarying quantities to expand the mental actions of Carlson et al.'s (2002) covariation framework. These forms of reasoning make finer grained distinctions in the "Quantitative Coordination" level of covariational reasoning. Distinctions made between these forms of reasoning might provide insight into how students could begin from informal reasoning to transition to more formal reasoning about average and instantaneous rate of change.

A Brief Overview of the Covariation Framework (Carlson et al., 2002)

Consideration of undergraduate and beginning graduate students' responses to tasks involving recognizing and characterizing how changes in one variable affected change in another variable (Carlson, 1998) led to the development of a covariation framework. The covariation framework (Carlson et al., 2002) provides a continuum of mental actions supporting five levels of covariational reasoning, with each level increasing in sophistication: Coordination, Direction, Quantitative Coordination, Average Rate and Instantaneous Rate. Researchers infer underlying mental actions from certain behaviors associated with each level of covariational reasoning. Classifying a student as reasoning covariationally at a particular level means that the student is able to perform mental actions supporting not only that level, but also all preceding levels of covariational reasoning (Carlson et al., 2002).

For the purposes of this paper, I focus on the Quantitative Coordination (QC) and Average Rate (AR) levels. The QC level supports the mental action of coordinating an amount of change in one quantity with the change in another quantity (Carlson et al., 2002). For example, a student who related amounts of change in volume to changes in height would provide evidence of reasoning at the QR level. The AR level supports the mental action of coordinating an average rate of change in one quantity with uniform change in another quantity (Carlson et al., 2002). For example, a student who related the rate of change in volume with respect to height to uniform changes in height would provide evidence of reasoning at the AR level. In a study of college calculus students, Carlson et al. (2002) found that even after students took a course focusing on rate and varying rate, students consistently applied covariational reasoning at the QC level, but

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not at the AR level. Further explication of the QC level of covariational reasoning might help to account for variation in the students' reasoning and suggest whether or not students' reasoning might advance to levels of Average (AR) and Instantaneous Rate (IR).

A key distinction between the QC and AR levels is the consideration of an amount of change (QC) versus the consideration of a rate of change (AR). In this paper, I provide two distinct forms of QC level reasoning that seem to support the addition of finer-grained mental actions to the covariation framework. These additional mental actions further explicate what it could mean to coordinate an amount of change in one quantity with change in another quantity.

Two Forms of Reasoning about Amounts of Change in Covarying Quantities

In this section I articulate both forms of reasoning, providing empirical support for each. I draw on three secondary students' (Austin, Jacob, and Hannah—names are pseudonyms) work on a task relating the typical high temperature of a city to the day of the year (see Fig. 1). Austin and Jacob were 11th graders enrolled in a Precalculus course and Hannah was a 10th grader enrolled in a Geometry course. The task required students to investigate how the typical high temperature varied as the day of the year varied. Each student worked on the task during an individual clinical interview (Clement, 2000), for which I served as the interviewer.

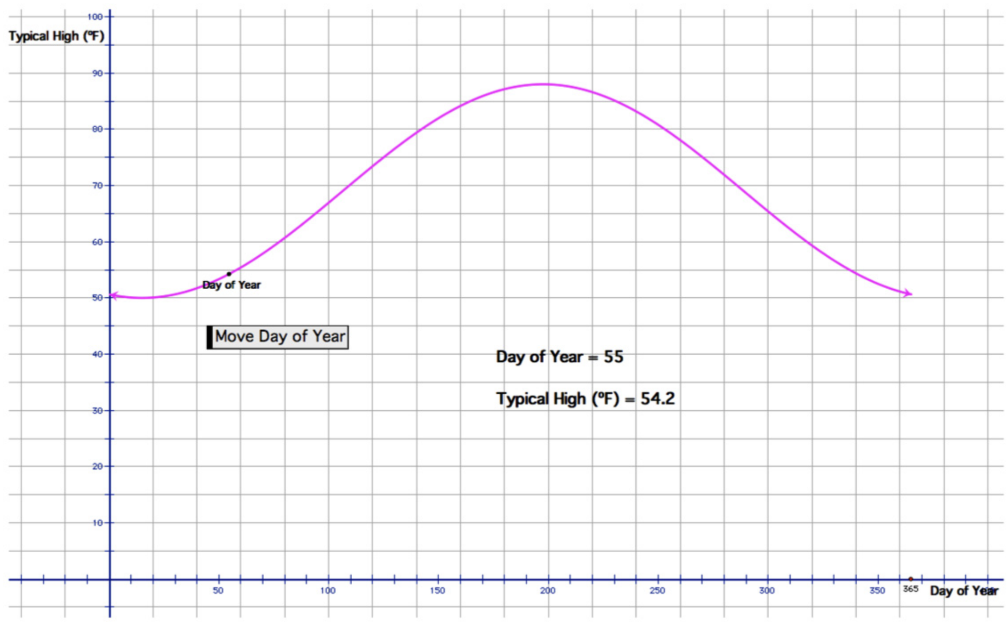

Figure 1: Dynamic Cartesian graph

The task incorporated a dynamic Cartesian graph (see Fig. 1) created using Geometer's Sketchpad Software (Jackiw, 2001). A student interacting with the graph could click and drag on the active point or press one of the animation buttons. As the day of the year changed, the corresponding typical high temperatures changed accordingly. As part of this task, I asked each student to use the graph to make a prediction about how the typical high temperature would continue to increase or decrease as the day of the year changed. Because the interviews were semi-structured, the actual prompt varied from student to student based on his or her individual work.

I employ an actor-oriented perspective (Lobato, 2003) when investigating students' reasoning about covarying quantities. By quantity, I mean an individual's conception of a "quality of an object in such a way that this conception entails the quality's measurability" (Thompson, 1994, p. 184). For example, a student could conceive of area as a quantity measuring an amount of flat surface being covered. By covarying quantities, I mean quantities that are changing together. For example, as a square is being enlarged, its side length and area are varying together. Drawing on students' explanation, written work, and gestures, I make claims about the mental actions involved in students' reasoning.

Changing Simultaneously and Independently

In the excerpt that follows, Austin used amounts of change in temperature and days to make claims about how the decreasing temperature is changing as the day of the year varied. When Austin used the word slope, he was referring to an association of an amount of days with an amount of degrees.

Interviewer: And when it decreases, if you had to describe for me, as it's going along, how is it decreasing as it's going along?

Austin: It just starts, like it's kind of rounded, or it's going more days for the temperature. It's kind of staying hot for a while and then once it starts to get close to say two hundred forty, two hundred thirty days, then it starts to decrease pretty much at that same constant rate as the other side as it increased.

...

Interviewer: And so, when you talk to me about decreasing, can you tell me what's decreasing?

Austin: The temperature is decreasing with the amount of days you go on from that top two hundred days.

Interviewer: So in the top here, how is that temperature decreasing?

Austin: From day two hundred to my line there *[longest horizontal segment shown in Fig. 2]*, it's close to about two hundred fifty, so in fifty days it's decreasing about seven degrees, which isn't that much. I'll write that down. It's fifty degrees in seven days there. *[Writes* $\frac{\text{50 degrees}}{\text{7 days}}$*]*

…

Interviewer: So suppose I were to ask you to consider the interval between day two hundred and day two twenty. How do you think that change would compare to this fifty days and seven degrees?

Austin: I'd say it'd be, it would change a little less because there's more or, there's less of a slope in those twenty days compared to that section there.

Interviewer: Can you show me? You can use the card *[Austin had been using a note card as a straightedge]*, or just show me what you mean by less.

Austin: You could just say like if I drew a line here, *[Draws in the upper left set of horizontal and vertical segments shown in Fig. 2]* it's changing a little, a lot less than compared to that. *[Draws in the lower left set of horizontal and vertical segments shown in Fig. 2.]*

Interviewer: And how does that affect the, how does that relate to the changing temperature?

Austin: It's just going to have a steeper slope, which means the more days, or the least, the lesser amount of days, compared; it takes for the temperature to drop a certain amount.

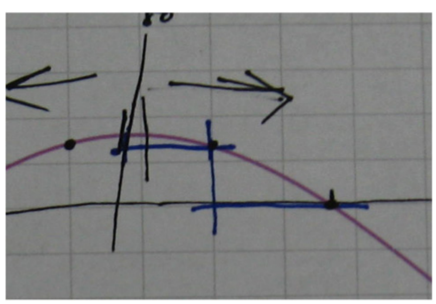

Figure 2: Line segments Austin drew to represent the changing amounts of temperature and days

To determine how the temperature might continue to decrease, Austin specified an interval of days and then compared the amount of change in temperature to the amount of change in days. He determined particular numeric amounts of change because he could compare the lengths of horizontal and vertical segments. With either specifying or not specifying numerical amounts, he used an interval, determined

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amounts of change in each quantity, and compared those amounts of change in the interval. Although not included in this excerpt, he did use division to compare the amounts of change in temperature and days. However, even when he used division, he interpreted the result as an amount of days per one degree of change in temperature, thereby preserving both individual quantities. Using Carlson et al.'s (2002) covariation framework, Austin was reasoning at the QC level, because he related amounts of change in covarying quantities.

Austin's reasoning shared similarities with Jacob's reasoning. In the excerpt that follows, Jacob determined an average rate of change in temperature per day for a five-day interval. He chose other five-day intervals that he predicted might have the same average rate of change, and then calculated the average rate of change on those intervals to make comparisons.

Interviewer: So if you were to determine an average change per day between days one ninety and one ninety-five, how would you figure that out, between one ninety and one ninety-five?

Jacob: Okay, well I'd take um, minus one ninety and I'll just do one ninety-five. Day one ninety-five has the high of eighty-seven point eight-nine, nine eight, (87.98) and one-ninety is a change of eighty-seven. Er, it doesn't have a change it has a temperature of eighty-seven point eight four (87.84). So to find change, ninety-eight minus eighty-four is point one four (0.14) that is for five days worth. So I would take point one four (0.14) divided by five to find the change in days, like per day so it changes point zero eight, two eight per day (0.028).

…

Interviewer: Are there any other time periods on the graph when you might expect an increase of point zero two eight (.028) degrees per day?

Jacob: Uh huh. Whenever, I'll go back to the beginning. Um, I'd say maybe somewhere around here. We'll say, we'll make it nice and make it forty. We'll try this.

Interviewer: Can you tell me why you picked this day?

Jacob: I just thought it looked like it wasn't moving up much.

Interviewer: And can you tell me how you determine if something looks like it's not moving up much?

Jacob: Um, yeah, it moves over a lot more than it moves up, so it means that it is not getting that much hotter as the days go on. But since it's curved inwards instead of outwards, I don't know if that is going to affect it, but I'm just going guess and write it down. Day, I would write day for the rest and the high was fifty-one point seven four (51.74). Day thirty-five, fifty-one point one (51.1), point six four (.64) difference for five days that is a lot bigger than this, point six four (.64) divided by five is somewhere around, yeah, point one two eight (.128) so that is a lot bigger I was wrong then, I'll go five more days, I hope so, I will be right this time.

Interviewer: Why are you moving left?

Jacob: Because if I went right it's getting greater, the intervals between each five days is getting bigger, because earlier I forget where I said it, yeah, here, it is moving up by about six point two degrees (6.2) every twenty days. … Six point two (6.2) divided by twenty, about point three one (.31), and up here it is just point zero one five (.015), so I don't, I don't see what's the point of even trying to go up because I know it is just going to get greater. So I will try, what day is this, thirty, fifty point, fifty point six one (50.61)… Fifty-one point one four (51.14) minus fifty point six one (50.61), point five three (.53). I don't know what that was—and so that's for five days so divide that by five so per day it changes point one zero six (0.106), that's still not even close. Let's go all the way back to the beginning day, it starts at day one and day six, I'll make another chart. How many do I have now, five? Yeah. Day one, day six, we have fifty point five three (50.53). Day six, fifty point two two (50.22), difference of, I am just going to use the calculator because I know what I want to say, point three one (.31) divided by five, point zero six two (.062). So I was wrong, we are probably not going to have a change like this. But that is kind of close, I guess, but that is as close as it is going to get. It just gets bigger and bigger as it is going, until it gets up to the top.

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To determine how the average rate of change in temperature given a five-day interval might compare to 0.028 degrees per day, Jacob calculated the average rate of change in different five-day intervals. As indicated by his comment about being “curved inwards instead of outwards,” he identified curvature as a physical attribute of the graph. He could use the shape of the graph to make some informed choices about where to begin his calculations. However, he was not able to use curvature to make sense of the varying average rate of change in temperature per day because his focus was on the results of his calculations. When his calculations did not support his hypotheses, he assumed that it was not possible to have another interval with the same average rate of change in degrees per day. Using Carlson et al.’s (2002) covariation framework, Jacob was reasoning at the AR level, because he considered the rate of change of temperature with respect to time for equal amounts of time (five-day intervals).

Together, Jacob and Austin’s responses provide empirical support for reasoning about covarying quantities as changing simultaneously and independently (see also Johnson, in press). This way of reasoning involves the simultaneous varying of quantities such that both are changing in tandem. Using this form of reasoning, a student could compare amounts of change in one quantity with amounts of change in another quantity in uniform or nonuniform intervals. A student could also use this way of reasoning to compare average rates of change in one quantity with respect to another quantity in uniform or nonuniform intervals. When using this form of reasoning, a student begins by forming intervals. In doing so, a student can compare amounts of change (or average rates of change) across intervals. Comparing variation in amounts of change in an interval would not be the student’s goal. Instead, the student’s goal is to find an amount of change (or average rate of change) in an interval, making varying change in the interval irrelevant.

Changing with Respect to Another Quantity

In the excerpt that follows, Hannah attended to variation in the intensities of increases and decreases in typical high temperature with respect to changes in amounts of days. Her reasoning stands in contrast to Austin’s and Jacob’s because she did not work from calculations to make claims about changes in the typical high temperature. Instead, she used descriptors such as “increases are increasing,” “steady increase,” and “increase its decrease” to indicate the variation in the intensity of an increase or decrease.

Interviewer: And so if you were to take a look over the whole year and talk to me about when the temperature, the typical high is changing the most or the least?

Hannah: The typical high changing the least would be like at the peak *[Makes a circling motion around the maximum of the graph shown in Fig. 1]* like near the one hundred ninety-seventh day, but like the least, or the most change would be around right here *[Motions to the part of the graph near day 60]*, like where the steady increase is going *[Slides her finger along the graph until about day 120]*, and like same on the other side, like around in there. *[Motions to the part of the graph near day 300.]* The peak is more like the least change.

Interviewer: And if you also had to talk about a range of days, and you talked about increasing increases,

Hannah: Mhmm.

Interviewer: When do you think, does it seem like those increases are increasing?

Hannah: Um, it looks like the increases are increasing right here *[Motions to the part of the graph between days 60 and 120.]* and then like the increases decreasing would be up closer to the point *[Referring to the active point which is on day 197]*.

Interviewer: When does it seem like the change happens from increasing increases to decreasing increases?

Hannah: It seems like it really changes before the steady increase. It’s where the increase increases and after the steady thing is where it starts to change to decreasing the increase.

Interviewer: And what about the decreases?

Hannah: The decreases is pretty much the same, like as the increases, except this is where *[Points to the part of the graph to the right of the maximum]* it starts to decrease its increase, or decrease its decrease, or no, increase its decrease, so that the other side towards the end *[Points to the right* *most portion of the graph]* would be where it's, the smaller decreases come.

Interviewer: Could you explain to me increase its decrease, just to make sure I'm understanding how you are thinking about these things?

Hannah: Like for, on the decrease side, around, like right after the point *[the maximum]*, like where the highest high is. Right after that the decrease is larger than what's after it. So the decrease starts off bigger and then as it goes on the decrease gets smaller. And then it goes into that steady one and then eventually the steady one goes smaller.

To determine how the intensity in the increases and decreases might vary, Hannah drew on the curvature of the graph to make claims regarding the intensity of the change. Hannah's work extends beyond noticing a physical attribute of the graph, because she could use an attribute (curvature) to make claims about variation in the increases and decreases in amounts of temperature. Using Carlson et al.'s (2002) covariation framework, Hannah was reasoning at the QC level, because she related amounts of change in covarying quantities. Although she attended to variation in the intensity of increases and decreases, she provided no evidence of considering average rate of change in temperature with respect to change in days.

Hannah's response provides empirical support for reasoning about covarying quantities such that one quantity changes with respect to changes in another quantity (see also Johnson, 2012). Using this way of reasoning, a student could vary one quantity (using uniform or nonuniform increments) and investigate how another quantity is changing with respect to that variation. Unlike a student reasoning about covarying quantities as changing simultaneously and independently (e.g., Austin & Jacob), a student reasoning about covarying quantities such that one quantity changes with respect to changes in another quantity (e.g., Hannah) does not necessarily form intervals to determine and compare amounts of change.

Expanding Carlson et al.'s (2002) Covariation Framework

Reasoning about covarying quantities such that one quantity changes with respect to changes in another quantity supports students' consideration of variation in intensity of quantity indicating a relationship between varying quantities. At the heart of this way of reasoning is the coordination of the covarying quantities such that one quantity is changing with respect to another quantity. In contrast, reasoning about covarying quantities as changing simultaneously and independently supports students' linearization of nonlinear situations, but does not support students' consideration of variation in intensity of a rate of change in a single interval. As evidenced by Jacob's work, reasoning about covarying quantities as changing simultaneously and independently could support covariational reasoning at the AR level. However, it seems unlikely that a student's mental actions would support reasoning about instantaneous rate of change.

I propose that the Carlson et al.'s (2002) covariation framework be expanded to account for students' reasoning about covarying quantities as changing simultaneously and independently (e.g., Austin & Jacob) or about covarying quantities such that one quantity changes with respect to changes in another quantity (e.g., Hannah). Using the current framework, Hannah and Austin were both reasoning at the same level (QC). However, these students were coordinating amounts of change in covarying quantities in very different ways. Making distinctions between the ways in which students coordinate amounts of change in covarying quantities can create two paths to the subsequent levels of Average (AR) and Instantaneous Rate (IR). Table 1 indicates two distinctions (Type 1 and Type 2) in the QC level of the covariation framework.

Table 1: Expanding the Covariation Framework

Level of Covariational Reasoning	**Mental Action**	**Behaviors**
Quantitative Coordination: *Existing*	"Coordinating each amount of change of one variable with changes in the other variable" (Carlson et al., 2002, p. 357)	• "Plotting points/constructing secant lines" • "Verbalizing awareness of the variable amounts of change of the output while considering changes in the input" (Carlson et al., 2002, p. 357)
Quantitative Coordination *Expansion: Type 1*	Coordinating amounts of change in quantities such that the quantities are varying simultaneously and independently	• Specifying intervals (uniform or nonuniform), determining amounts of change in those intervals, and comparing those amounts of change • Using amounts of change to make claims about covarying quantities
Quantitative Coordination *Expansion: Type 2*	Coordinating amounts of change in quantities such that change in one quantity depends on change in another quantity	• Allowing one quantity to change with respect to another quantity • Describing variation in the intensity of change in covarying quantities

By making these distinctions in the QC levels, students' transitioning to more advanced levels of covariational reasoning might be more closely examined. Students engaging in QC Type 1 covariational reasoning seem likely to advance differently to the levels of AR and IR than would students engaging in QC Type 2 covariational reasoning. For example, Jacob reasoned in a way consistent with QC Type 1 and provided evidence of reasoning at the AR level. To extend to the IR level of covariational reasoning, a student could begin by shrinking the interval on which average rate of change is being determined. In Jacob's work on the task, he was able to shrink the interval when prompted. However, his goal was not to shrink the interval because his focus was comparing average rates of change in different intervals. In contrast, it made sense for Hannah to consider smaller intervals because for her the change in temperature was dependent on the change in the day of the year. Future research might investigate how students using these different types of QC covariational reasoning advance to AR and IR levels of covariational reasoning.

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