WHAT SENSE DO CHILDREN MAKE OF NEGATIVE DOLLARS?

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We interviewed 40 students in Grade 7 to investigate their integer reasoning. In one task, children were asked to write and interpret equations related to a story problem about borrowing money from a friend. Their responses reflect different perspectives concerning the relationship between this real-world situation and various numerical representations. We identify distinct ways in which integers were used and interpreted. All of the students solved the story problem correctly. Few thought about the story as involving negative numbers. When asked to interpret an equation involving negative numbers in relation to the story, about half related it to the story in an unconventional fashion, which contrasts with typical textbook approaches. These findings raise questions about the role of money and other contexts in integer instruction.

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Introduction

The use of contexts is common in integer instruction. In a review of fifth and sixth grade textbooks adopted by the state of California, we found that 94% of these used the context of money in instruction concerning integers. Elevation (89%) and temperature (89%) were also popular contexts (Whitacre et al., 2011). There are conventional ways in which textbook authors relate these contexts to the integers and to integer arithmetic. These conventions may or may not jibe with children's intuitions. In interviews with 40 seventh graders, we investigated how children made sense of a story problem concerning borrowing money from a friend, and how the children saw this context as related (or not) to various equations. We report on children's ways of reasoning about the relationship between the context and the equations. These new findings concerning children's reasoning about integers suggest implications for integer instruction.

Theoretical Framework: A Children's Mathematical Thinking Perspective

We approach this study from a children's mathematical thinking perspective. In the tradition of Cognitively Guided Instruction (CGI), we value children's mathematics. We take seriously the nature of that mathematics, even if it is incorrect from an expert perspective. We endeavor to see the mathematics through children's eyes in order to better understand the sense that they make (Bishop et al., 2011; Lamb et al., in press). We do this because the ultimate goal of our research is to find ways to better support children's learning of mathematics (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Franke, & Levi, 2003; Empson & Levi, 2011).

A Distinction Informed by the History of Mathematics

There is a somewhat subtle distinction that is relevant to the analysis presented in this paper. As a matter of background, the notion of a negative number was controversial historically. Many mathematicians resisted the idea of negative numbers because numbers were associated with magnitudes, and magnitudes less than zero seemed nonsensical. During the 13th through the 18th centuries, many Western mathematicians used negative numbers in algebra, although these remained hotly debated and

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were not generally regarded as legitimate numbers (Gallardo, 2002; Hefendehl-Hebeker, 1991; Henley, 1999).

Taking account of the history of negative numbers, it would be tempting to say the following: "Positive numbers existed before negatives. Furthermore, even after negatives came into use, it took a long time for them to become understood and accepted as legitimate numbers." We would take exception with one aspect of this account—the use of the word *positive*. Prior to the advent of negative numbers, there was no such thing as a positive number. Positive and negative are opposites. Their meaning derives from the contrast between the two. Before the notion of a negative arose, there were simply numbers. In keeping with the language that many children have used in our interviews, we refer to these as *regular numbers.*

Consider the integers. It is commonplace today to refer to the positive integers and the natural numbers as one in the same—at least among mathematicians or in mathematics classes at the middle school level and above. Historically, however, the natural numbers predate the integers. More importantly, there is a conceptual distinction between these sets. The positive numbers are signed numbers. The natural numbers are not. Put another way, signed numbers may be thought of as *directed magnitudes*. That is, they convey two distinct pieces of information, direction (sign) and magnitude (absolute value). The positive numbers have this property. The natural numbers, by contrast, have only magnitude.

This distinction is relevant to the experience of children learning mathematics today. For the first so many years of a child's life, numbers are not signed. Children learn to count with natural numbers. At some point, they learn about zero. Later, they encounter nonnegative rational numbers, which they come to know as fractions and decimals. Typically, a child's introduction to negatives (and positives) comes after all this. In fact, we have found that many elementary children who have some familiarity with negative numbers have never heard of positive numbers. When they learn about integer arithmetic, which typically occurs in middle school, children are told that the regular numbers are actually positive. However, there is a sense in which this distinction remains salient. The distinction between positive numbers and regular numbers plays a part in the analysis that we present.

Methods

We interviewed 40 children in Grade 7 during the spring of 2011. The interviews were conducted at public schools in an urban area in California. All Grades 5, 6, and 7 mathematics textbooks adopted by the California Department of Education contain material concerning integers and integer arithmetic. Thus, we expected that the students we interviewed all would have received substantial instruction on integers. Indeed, we know from the interviews that all 40 of the children were familiar with negative numbers and were able to solve (at least some) problems involving integer arithmetic.

The interviews consisted of a range of tasks, including open number sentences, number comparisons, and story problems. Interviews were conducted at the children's school sites, during the school day. Interviews were videotaped and typically lasted between 60 and 90 minutes.

Task: The Money Problem

In this report, we focus on the following story problem:

> *Yesterday, you borrowed $8 from a friend to buy a school t-shirt. Today, you borrowed another $5 from the same friend to buy lunch. What's the situation now?*

The interviewer would often clarify the question by asking, "Do you owe your friend money? Does your friend owe you money? How much money?" Students were asked to solve the problem and to explain their thinking. They were also asked to write a number sentence that would represent the problem, including its solution, and to explain how this equation related to the story. Students were then asked if they could write additional equations that would also represent the story. Next, they were presented with three equations, one by one. They were told that these had been written by other children to represent the same story. Students were asked to tell whether or not they thought each equation matched the story and to explain why. The equations shown to students were the following:

i. -8 + -5 = -13
ii. -8 – 5 = 13
iii. 8 + 5 = 13

Order of presentation varied. Typically, if one of equations i–iii matched an equation that the child had written, this one was shown first.

Analysis of Children's Responses

We first analyzed a subset of the data qualitatively, focusing on children's solution strategies and underlying ways of reasoning. We used principles of grounded theory (primarily the constant comparative method) to identify emergent, distinguishing themes in students' reasoning (Strauss & Corbin, 1998). Once a set of codes was generated that fit this subset of the data, these were used to code the remainder of the data. In particular, our analysis of children's responses to the story problem revealed an interesting theme, that of *perspective*. This is a category of codes. Within that category, we identify three ways of reasoning with regard to perspective: *conventional*, *unconventional*, and *perspectiveless*. These three ways of reasoning were used to code the responses of all 40 children. These apply to children's explanations of their own equations, as well as to their interpretations of equations i, ii, and iii.

As a reliability check, 25% of the data was double-coded. For 10 of the 40 children's responses, two researchers independently coded those responses using the perspective codes. Coders agreed on the ways of reasoning of 9 of the 10 children (90%). The one instance of a disagreement was a case of coder error, and it was corrected. We also coded children's responses to the story problem as correct or incorrect, and we recorded the equations that they wrote. We then tabulated the results.

We report here on the percentage of children who solved the story problem correctly, the percentages of children who wrote certain equations to represent the story problem, and the percentages of children who interpreted equations involving negatives from a conventional or an unconventional perspective.

Results

First, we describe the three different ways of reasoning. Then we provide specific examples of each way of reasoning. Finally, we report the results of our analysis of the responses of all 40 children who were interviewed.

Ways of Reasoning

We describe the ways of reasoning that were identified.

Perspectiveless. The child sees regular numbers as an appropriate representation of amounts of money. Thus, 8 + 5 = 13 would appropriately represent the situation because 8 represents $8, 5 represents $5, and 13 represents $13. The fact that this was money borrowed from a friend is relevant only in that the answer of 13 refers to a total amount of money borrowed/owed. However, the same equation could just as well represent the situation from the lender's side. In either case, $8 plus $5 is $13. The numbers are being used here only to communicate magnitude, not direction (of owing).

Unconventional perspective. The child views positive numbers as representing money that the *borrower* gained, even though the money was borrowed. From this perspective, 8 + 5 = 13 matches the story because 8 represents $8 given to me by a friend, 5 represents an additional $5 given to me by that friend, and 13 represents the total of $13 that I acquired. For children reasoning from this perspective, -8+ -5 = -13 does not match the story from the borrower's side, but it could match the story from the lender's side because the lender lost money by lending it.

Children whose responses were coded as reflecting an unconventional perspective were taking perspective into account. They reasoned about the appropriateness of using positive or negative numbers to represent the situation. Negative numbers had meaning for these students, albeit an unconventional meaning. We use the term *unconventional* because, in our review of textbook approaches to integer instruction, we did not find negatives used in this way. Furthermore, the unconventional perspective contrasts directly with the conventional perspective, which is typical of textbook treatments.

Conventional perspective. The child views negative numbers and subtraction as representing debt or net loss. For example, -8 + -5 = -13 is seen as matching the story about borrowing money from a friend because -8 represents a debt of $8, -5 represents an additional debt of $5, and -13 represents the total debt of $13. From this perspective, positive numbers would be used to represent the lender's situation. That is, 8 + 5 = 13 would not describe the situation from the borrower's side, but it would describe the situation from the lender's side.

Children's Responses

Below, we present examples of particular children's responses that illustrate these distinct ways of reasoning.

Perspectiveless. Elisa wrote 8 + 5 = 13 as her equation. Like the other children who wrote this equation, her explanation was perspectiveless:

Interviewer: Okay. And can you explain to me how this equation matches the story?
Elisa: Okay. So, yesterday I borrowed eight dollars from my friend to buy a school t-shirt. So, I have eight dollars from my friend [points at "8" in equation]. And then today I borrowed five from the same friend, and I bought a lunch. And then plus five that I borrowed from her [circles "+ 5" in equation] equals thirteen dollars that I borrowed from her [circles "13" in equation]. And what's the situation? I owe her thirteen dollars.

Elisa made sense of the story and was well aware of who owed money to whom. However, this directional information was not conveyed in her equation. It belonged to what she knew the equation represented—how much money she owed her friend. The equation itself simply conveyed magnitude information: the sum of $8 and $5 is $13.

Unconventional Perspective. Tommy also wrote 8 + 5 = 13 to represent the story. When he was shown -8 + –5 = 13, he said that it would not work to describe the borrower's situation. Tommy's explanation is an example of unconventional perspective:

Interviewer: Can you read that one for me? [Interviewer reveals -8 + -5 = -13 on paper]
Tommy: Negative eight plus negative five equals negative thirteen.
Interviewer: Okay, and what do you think about that? Do you think that that describes the situation, or the story?
Tommy: Um, no. Because it's like, it's basically like saying that they owe you because it's like, it's like you're not taking any money. It's like they're taking your money. Because it's negative, which means it's like, it's kind of lower; it's lower than zero. Like, so then it's like they're owing you money, instead of you owing them.

Tommy interpreted negative numbers as indicating someone "taking" money. He viewed the situation from his side (as the borrower), and so he said that the equation would not fit because the negatives would mean that money had been taken from him, rather than given to him.

Conventional Perspective. Jen initially wrote -8 – 5 = x as her equation. At the interviewer's request, she later rewrote her equation, replacing x with a known number (-13). Her explanation conveyed a conventional perspective:

Interviewer: Okay, so tell me about what you wrote.
Jen: I wrote negative eight minus five equals x [pause] because, um, because if I borrow money, it's like I'm, I'm like losing money. No, it's not I'm losing money. It's like I'm, I'm borrowing someone else's money. So, on my side, it would be a negative. And then on another day, I'm also borrowing money, so that's also like a negative.
Interviewer: Okay. And what would the x be here?

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Jen: It'd be like how much money I borrowed in total.
Interviewer: Mm-kay. And could you rewrite this and replace x with the answer?
[Jen writes -8 – 5 = -13]
Interviewer: Okay. So, why did you write negative thirteen for the answer there?
Jen: Because negative eight minus five is negative thirteen.
Interviewer: And how does that relate to the story?
Jen: It relates to the story because, if I borrow eight and five, I owe them thirteen. So, it's like a negative thirteen dollars on my side.

Jen articulated a view of negatives as representing debt. For her, -8 represented $8 that she had borrowed, – 5 represented $5 that she had borrowed, and -13 represented the total amount that she owed to her friend.

Overall Results

All 40 of the 7th graders (100%) solved the story problem correctly. That is, they said something like, "I owe my friend $13." They were then asked to write one or more equations to represent the story. Of the 40 students, 33 (82.5%) wrote 8 + 5 = 13 as one of their equations. Usually, this was the first equation that they wrote. Often it was the only equation that they wrote. Rarely did students write equations involving negative numbers. Those who wrote multiple equations typically created variations that also involved natural numbers. The equation 5 + 8 = 13 was the most common second choice. In each case that a child wrote 8 + 5 = 13, the child's explanation for that equation was perspectiveless. The child talked about the numbers as representing amounts of money, and there was no evidence that he or she intended to convey information about the direction of borrowing/owing with the signs of the numbers. These children were using regular numbers.

Only 8 of the 40 students (20%) wrote an equation involving negative numbers. Six children wrote -8 + -5 = -13, and two children wrote -8 – 5 = -13. Children's explanations for these equations naturally addressed the issue of perspective since the children had made a choice to use negative numbers. Thus, only 20% of the children took perspective into account in writing an equation to represent the situation.

The children were then shown equations i, ii, and iii. We report here on the perspective reflected in their interpretations of equations i and ii (-8 + -5 = -13 and -8 – 5 = -13). Often children's responses explicitly addressed perspective for either i or ii but not both. In cases where perspective was explicitly addressed for both, the perspective was always consistent. For these reasons, we group children's responses to i and ii. Of the 40 children who were shown equations i and ii, 19 of them (47.5%) articulated a conventional perspective. They interpreted the negative numbers in the given equation as representing debt. For them, the given equation was appropriate for describing the borrower's situation. An additional 19 children (47.5%) expressed an unconventional perspective in interpreting one or both of equations i and ii. These children viewed negative numbers as representing money lost or taken away. For these children, the given equation described the lender's situation. Two of the children (5%) were not given any perspective code for i or ii. It was not clear that they had any way of interpreting negative numbers with respect to the context.

Discussion

We have identified three distinct ways in which children reason about equations like 8 + 5 = 13 and -8 + -5 = -13 in relation to a story about borrowing money. Our analysis has involved two key ideas that arose in children's mathematical thinking. The first is the distinction between regular numbers and positive numbers. When children wrote 8 + 5 = 13 to represent the situation, they were using regular numbers. They viewed this equation as related to the context in terms of the numbers and the operation involved. The children were aware that the number 13 represented $13 borrowed from a friend; however, information about the direction of borrowing/owing was not contained in the equation.

The second key idea involves another distinction, and it applies only to those children whose mathematical worlds include positive and negative numbers *and* who see these as related in some way to the context of borrowing/owing money. Our analysis of the responses of children like these revealed two distinct perspectives, conventional and unconventional. Of the 38 children who were able to interpret negative numbers in relation to the story, exactly half of them expressed a conventional perspective. They interpreted the negative numbers in the given equations as representing debt, or money owed by the borrower to the lender. The other half of the children expressed an unconventional perspective. They interpreted negative numbers in the opposite fashion, as expressing money lost by, or taken away from, the lender. Both groups of children consistently viewed the equations as making sense for describing the situation of one person in the story (either the borrower or the lender) and not the other. In this respect, the two perspectives are incompatible. Their interpretations disagree with one another.

Implications

Story problems in real-world contexts are widely used in integer instruction, and stories involving money are among the most common of these. The reason we have used the terms *conventional* and *unconventional* to describe students' reasoning in relation to these contexts is that the conventional perspective is typical of the way that positive and negative numbers are used to represent debt in mathematics textbooks. That is, the conventional perspective is the accepted perspective of the mathematical community. It is also reflected in the standard notation used in bank statements, utility bills, and so on. Negative numbers are used to denote a debt or debit; positive numbers are used to denote a deposit or credit. Thus, it is noteworthy that this convention is not consistent with the reasoning of many children. In particular, our review of textbook approaches to integer instruction suggests that the vast majority of 7th graders have encountered money contexts in their instruction concerning integers. Yet approximately half of the 7th graders that we interviewed interpreted the relationship between integers and money from the unconventional perspective, which contrasts with the interpretation reflected in the textbooks.

These findings raise questions concerning the roles of contexts such as money in integer instruction. Our group has begun to question what it means to make sense of integers and integer arithmetic. Often making sense in K–8 mathematics seems to be defined in terms or relating numbers and operations to quantities in the world. At least in the case of the integers, we believe that this notion may be in need of revision. Certainly, a mature understanding of integers includes the ability to relate them to quantities in the world. However, this does not entail that reasoning about real-world quantities like money should serve as a *source* for children's mathematical intuitions. On the contrary, we can imagine children developing a deep, purely mathematical understanding of the integers. This understanding could then be superimposed upon real-world situations, in the way that we do as mathematically literate adults.

We hope that the distinctions that we have discussed may be useful to both researchers and practitioners who are interested in the teaching and learning of integers and of integer arithmetic. In particular, sensitivity to the distinction between regular numbers and positive numbers can inform the language that we use with students and the care that we take in introducing them to the notion of signed numbers. Likewise, sensitivity to the issue of perspective can inform instructional decisions. It reminds us of the importance of eliciting the details of students' thinking and of having explicit discussions of different ways of reasoning.

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