

## CONNECTIONS ACROSS REPRESENTATIONS IN STUDENTS' GROUP DISCUSSIONS OF A NON-ROUTINE PROBLEM

**William Zahner**

Boston University

wzahner@bu.edu

*This research report examines how two groups of bilingual algebra students made connections among representations while solving a non-routine generalization problem. Using a socio-cultural orientation to mathematics learning, together with a semiotic lens on students' joint mathematical activity, this report details the type of connections among representations each group of students made as they solved the problem. Follow-up analysis shows that some connections afforded making more productive conclusions while other connections may have constrained the groups' solution processes. Finally, analysis of change across time reveals that the initial connections made by each group persisted across six weeks, despite intervening instruction that suggested other possible connections to solve the problem. The conclusion contains implications for researchers and practitioners.*

**Keywords:** Generalizing; Connecting Representations; Collaborative Learning; Algebra and Algebraic Thinking

This paper reports on how two groups of bilingual algebra students made connections among multiple representations of an algebra problem (a story, diagrams, a table, equations, and a graph) while solving a non-routine problem about generalizing a linear relationship. This analysis addresses the continuum of student learning in school mathematics. Making connections—whether across presentations, among mathematical concepts, and/or between situations and mathematical representations—is a critical component of mathematical understanding (Hiebert & Carpenter, 1992; National Council of Teachers of Mathematics, 2000; National Research Council, 2001; Presmeg, 2006). Given the centrality of connections in students' conceptual understanding (Hiebert & Carpenter, 1992), prior research on how students learn about and reason with linear functions has examined the ways that students connect or coordinate representations (e.g., Brenner et al., 1987; Lobato, Ellis, & Muñoz, 2003; Moschkovich, Schoenfeld, & Arcavi, 1993; Presmeg, 2006; Radford, Bardini, & Sabena, 2007).

This paper extends prior research on mathematical connections by examining the affordances of different connections made by two groups of students as they solved a non-routine generalization task. The data for this empirical report are from a study that investigated how bilingual students learned to reason about the rate of change of linear functions through engaging in peer discussions. This research report focuses on findings related to the question, How did each group of students use and connect multi-semiotic tools to solve generalization questions during peer discussions of a non-routine algebra problem? Initial answers to this question led to a follow up analysis of the affordances of the different mathematical connections each group made and the relative persistence of the connections made by each group.

The primary finding, which is outlined in more detail below, is that each group connected two or more representations as they reasoned through the given task, but there were important differences in the combination of connections that each group relied upon. The connections each group made afforded (and at times constrained) making accurate generalizations. Moreover, the combination of connections each group discussed remained mostly stable over a time period of six weeks.

### Framework and Prior Research

This study is grounded in a sociocultural approach to thinking and learning (Vygotsky, 1978; Wertsch, 1998). Under this approach, learning mathematics may be examined as a process of appropriating culturally shared tools for engaging in mathematical activities (Forman, 1996; Moschkovich, 2004; Rogoff, 1990). For example, in the domain of school algebra, learning is evidenced by the increasingly skillful use of tools for algebraic problem solving—where tools include things such as mathematical

inscriptions, standard algorithms, and mathematical discourse practices. This framework acknowledges that the meaning of these tools for thinking is not static. For this reason, many researchers in this tradition often refer to *inscriptions* rather than *representations* (Sfard & McClain, 2002). This report uses the term *representations* to align with prior research and with standards documents in mathematics education, while retaining the notion that the meaning of representations is constantly under negotiation.

When examining whether and how students make connections, a critical question that arises is, *what* do students connect? This analysis focuses on the connections that these students made between the multiple semiotic resources available in the problem (a story, diagrams, a table, a graph, and questions) as well as representations constructed by the students (equations, numerical answers, alterations to the diagrams, et cetera). Evidence of connections is visible when students use or coordinate multiple representations (Hiebert & Carpenter, 1992) and this can be observed in their talk, gestures, and writing (Moschkovich, 2008; Radford, Bardini, & Sabena, 2007). For example, at one point in this study, one student referred to the problem story while pointing at a diagram as his group debated which numbers to fill in to a table of values. In this case, I argue that the students connected the story, the diagrams, and the table.

The content focus of this study is how students generalize linear relationships and reason about the slope of linear functions as a rate of change. While understanding the relationship between slope and rate of change is a critical topic in school mathematics, prior research suggests that many students struggle with this concept, and that the origin of these difficulties may lie in how students and their teachers use representations and computational procedures to reason about the slope of linear functions (Leinhardt, Zaslavsky, & Stein, 1990; Lobato, Ellis, & Muñoz, 2003).

In a review of the literature on student learning of mathematical functions, “making connections” was identified as a key component of fluent reasoning with mathematical functions (Wlimot, Schoenfeld, Wilson, Champney, & Zahner, 2011). The centrality of connections is also highlighted in the descriptions of mathematical discourses as multi-semiotic (O’Hallaran, 2003; Radford, Bardini, & Sabena, 2007). From a semiotic perspective, generalizing from particular cases requires seeing the particular (e.g., the first three terms of a geometric pattern) as representative of more than the particular (e.g., the  $n$ th term of the pattern). Therefore generalizing is intimately related to making connections (Radford, Bardini, & Sabena, 2007). This work extends prior work reported in Wilmot et al. (2011) by examining (a) how connections developed across time during group discussions, and (b) how the quality of connections mediated each group’s success in generalizing a linear relationship during a group discussion.

## Methods

### Setting and Participants

This study examined the mathematical reasoning of two groups of ninth grade students enrolled in a bilingual algebra class at a comprehensive high school in an agricultural region of California. Over 90% of the students at the school were Latino/a and 35% of the students were classified as English Learners. Seventy-seven percent of the school population was eligible for a free or reduced price lunch. The two groups were enrolled in a bilingual algebra class taught by an experienced teacher who has been recognized for her excellent teaching and for her skillful use of group work. Thirty percent of the students in the class spoke primarily Spanish, and the remaining students spoke both Spanish and English. The bilingual setting was chosen intentionally because it is a site where attention to language and meaning in mathematics was likely (Sierpinska, 2005).

The algebra curriculum focused on reasoning and problem solving in real-life contexts, and this data collection coincided with a unit focused on interpreting data and reasoning with linear functions (Fendel, Resek, & Alper, 1996). In consultation with the researcher, the teacher selected two focal groups of four students each. The groups were chosen to be representative of the class and each group had students with a broad range of prior mathematics achievement.

Group 1 consisted of two boys and two girls who all reported speaking Spanish at home, but who primarily spoke English in class. Two of the students in Group 1 moved to the U.S. from Latin America as

children, but they were classified as “Fully English Proficient” by their school at the time of the study. Group 2 included four students who were all recent immigrants from Latin America. All members of Group 2 were classified as English Learners. The members of Group 2 spoke mostly Spanish in class, and the teacher provided them with copies of the curricular materials in Spanish. Two members of Group 2 left the school halfway through the data collection, and two other Spanish-dominant students in the class replaced them in the group.

### Data Collection

The principles for data collection were derived from Moschkovich and Brenner’s (2000) naturalistic paradigm for research on mathematical thinking, as well as the microgenetic method for examining learning across time (Chinn, 2006). Following these principles, the data collection included six weeks of in-class observations as well as a series of three out-of-class group problem solving discussions. The out-of-class group discussions were designed to systematically document change across time in the students’ reasoning on non-routine problems that required using important concepts related to rate, slope and reasoning with linear functions. The in-class observations are regarded as naturalistic observations that reveal how the students’ mathematical reasoning developed in relation to ongoing activity (Moschkovich & Brenner, 2000). Due to length restrictions, this report focuses on the out-of-class group discussions.

Each group participated in three out-of-class discussion sessions: one near the start of the unit, one near the middle, and one a week after the unit was finished. The problems that the students solved during these discussions were adapted from previous research and piloted before the data collection. Each group worked on the same problem multiple times across the six weeks, allowing for direct comparisons of changes and similarities in the groups’ reasoning across time (Chinn, 2006). In the protocol for these discussions, the students were instructed to discuss each problem as a group, come to agreement, and write one agreed-upon answer on the group’s paper. These discussions were video recorded, and copies of the students’ final answers and scratch work were collected. The videos were transcribed with a focus on capturing the propositional content of the students’ talk, and gestures were included in the transcript when students made deictic statements. This analysis focuses on the groups’ discussions of one task, Hexagon Desks. This task was chosen because (a) each group discussed it thoroughly during each discussion session, (b) both groups appeared to come to consensus on their answers to this question, and (c) this question invited the most “real life” connections for the students.

Hexagon Desks asked the groups to construct a generalized linear relationship describing how many people could sit around a row of 1, 2, 3, and more Hexagon Desks arranged in a row. Figure 1 contains a copy of the problem in English (Group 2 received the problem in Spanish). Variations of this task have appeared on the National Assessment of Educational Progress, among many other venues. The type of questions and the order of the questions in Hexagon Desks were written to parallel questions from mathematics problems that the students completed in class (e.g., observe a numerical pattern, make a table, and then generalize).


The first question on Hexagon Desks asked students to complete a table showing the number of people who could sit at a row of 4, 5, 6, and 7 desks. Question 2 required the students to find how many people could sit at a row of 100 desks. Question 3 required generalizing the pattern for  $n$  desks. In Question 4, the students graphed points from the table on a given graph. Question 5 asked the students to imagine connecting the points with a linear function and to compute the slope of the linear function (the question was worded to sidestep the issue of discrete and continuous functions since that was not a topic of discussion in the students’ class). Question 6 asked the students to explain how the slope of the linear model related to the story about desks. Finally, Question 7 was an open ended question asking the students to explain how their answers would change if the desks were octagons rather than hexagons.

Some questions on Hexagon Desks demanded making at least one connection, while other questions invited, but did not require, making multiple connections. For example, Question 6 asked the students to explicitly connect the slope of the linear model back to the story about pushing desks together (e.g., “the slope is 4 because each new desk adds four new places at the row of desks”). This question required making a connection between two representations. In contrast, Question 1 invited, but did not require,


making connections because it could be solved without connecting representations by noticing the numerical pattern within the table.

Ms. West wants to know how many students can sit around a row of hexagon shaped desks.


If one desk is by itself then six students can sit around it.



If two desks are pushed together, then 10 students can sit at the table.



If three desks are pushed together in a row as shown below, then 14 students can sit together.



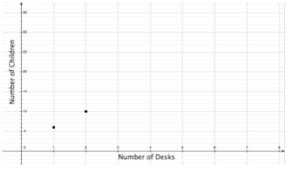
1. Fill in the following table for the number of students who can sit together for the number of desks pushed together in a row:

Number of Hexagon Desks	Number of Students
1	6
2	10
3	14
4	
5	
6	
7	

2. Imagine that 100 of the hexagon desks were pushed together in a row. How many students could sit around that row of desks? Show the work you used to find that solution.

3. If  $n$  hexagon shaped desks, were pushed together, then how many students could sit at the row of desks? Give your answer as a formula in terms of  $n$ .

4. Use the table you made in problem 1 to draw a graph showing the number of children who could sit at a row of desks.



5. If you connect the dots between points in the graph to make a line, what is the slope of that line? How do you know?

6. What is the meaning of the slope of the line *in terms of the problem about children sitting at desks*? Explain your answer in terms of the problem and using words and ideas that you know from math class.

7. What if  $n$  octagon-shaped desks were pushed together? How would this problem be different? How would it be the same? Explain your answer in as much detail as possible (you may use equations, tables, graphs, words, etc.).

**Figure 1: The task the students discussed**

## Analysis

This analysis relied primarily on the transcripts of the group discussions and the copies of the students' written work. However, the video recordings were used throughout the analysis process to clarify ambiguities in the students' talk and to document the students' gestures. The transcripts were divided into segments corresponding with each group's work on a particular problem in Hexagon Desks. For example, each group's talk about Question 2 was one segment in each transcript. Some segments were divided into sub-segments when the group discussed subparts of a question separately. For example, as Group 1 discussed Question 1 of Hexagon Desks, they engaged in several sub-discussions to decide which values to add in each cell in the table.

The first stage of analysis required documenting the connections among representations made by each group. Connections were coded by noting when each group made verbal, gestural, or written references to more than one representation during a particular segment in the transcript. For example, when completing the table of values on Question 1, Mateo in Group 1 explained why the net result of adding a new hexagon is adding four new spaces: "you add another one [hexagon] and nobody's gonna be sitting on that one." As he said "sitting on that one," he pointed to a vertical line at the intersection of two hexagons on Krystal's paper. With this utterance the group was coded as making a connection between the numerical table, the diagram, and the story about seating students at desks.

In addition to documenting connections, each group's final written answers were analyzed to examine whether the final result of the group's discussion was a correct response to each question. This analysis led to claims about the relative affordances of different connections made by each group. Finally, the connections made by each group and each group's agreed-upon final answers were compared across the three discussion sessions to analyze whether and how the groups' connections developed across time.

## Results

Each group made multiple connections as the students worked through the questions on Hexagon Desks. In general, Group 1 had longer discussions and they made connections among multiple semiotic resources as they solved each question, while Group 2 tended to be focus on pair-wise connections. For example, in the quotation from Mateo in Group 1 above, Mateo connected the table, the given diagrams,

and the story about seating students at desks. In contrast, when Group 2 discussed the same question on Hexagon Desks, they focused exclusively on the numerical pattern within the table. This is illustrated in the following excerpt from their first discussion of Hexagon Desks. (Note: in the transcript, comments are in double parentheses, while translations are in double parentheses and quotation marks).

1. Hector            Son seis catorce, son seria (“it’s six, fourteen, they are a series”)
2. Iris                Yo puse xxx de cuatro (“I put xxx by four”)
3. Hector            Dieciocho ((looks at Iris and points at Graciela's paper)) Dieciocho (“eighteen eighteen”)
4. Graciela        Por qué? (“why?”)
5. Hector            Dieciocho, son cuatro-- cada uno tiene cuat- (“Eighteen it is four, each one has four”) Este tiene seis, este tiene diez, y este tiene catorce ((pointing at the table on Graciela’s paper)). Cuánto es la diferencia? (“This one has six this one has ten, and this one has fourteen. How much is the difference?”)
6. Graciela        Cuatro. (“four”)
7. Hector            Son cuatro ((lifts up four fingers)) (“They are four”)
8. Graciela        Um
9. Hector            Entonces son dieciocho (“Then they are eighteen”)
10. Iris              Son dieciocho (“they are eighteen”)
11. Hector          Son veintidos(.) Son vientosis(.) Son treinta(.) (“They are twenty two, they are twenty six, they are thirty”)
12. Graciela        Ah hum ((nods her head up and down))

This trend in the way each group made connections held across all three discussions. Group 1 repeatedly made multiple connections among three representations for the problem: they referred to the given diagram, the story, and the table as they solved Questions 2, 3, and 7, and they discussed how these representations were related. Group 2 adopted a narrower focus, they consistently connected each question back to the numerical patterns from the table. For example, when Group 1 generalized the pattern for 100 hexagons (Question 2), they focused on the contributions of the top, bottom, and sides of the diagram of a chain of hexagons to calculate the answer of  $200 + 200 + 2 = 402$ . In contrast, each time Group 2 attempted Question 2, they attempted to generalize the numerical pattern using only the table. Their answers across all three discussions were  $100 \times 4 = 400$ , 150, and  $42 \times 10 = 420$  respectively, showing that they were not able to use this numerical pattern to successfully generalize to the hundredth case.

A first follow up analysis compared the connections across representations that the groups made with each group’s agreed upon final answers. While both groups made some connections, not all connections proved equally useful for solving the problem or reaching a generalization. Group 2’s responses to Question 2 show that their ways of focusing on the numerical pattern in the table was not useful for developing a generalization about how many students could sit at a row of 100 desks. By the third discussion both groups were able to use the graph to successfully compute the slope in question 5. Group 2 was able to correctly answer Question 6 (interpret the meaning of the slope) by noting that the slope of the linear model was 4, and they also used a connection between the linear function and the story to describe the meaning for the slope, saying that this slope was the same as the “add four” that results from adding an additional desk to the row of desks. Meanwhile, although Group 1 was able to calculate the slope using the graph, they did not describe the meaning of the slope in relation to the problem, thus there was no evidence that they were connecting representations to justify their response to the slope interpretation question. One possible explanation for Group 1’s difficulties interpreting the slope is that the net change of “add four” was not as readily apparent when focusing on the “adding and subtracting” action of adding a new hexagon on the end of the diagram. Thus, the different connections made by each group provided different affordances for justification and for generalization.

Finally, the second round of follow-up analysis examined how the connections made by each group shifted as they solved Hexagon Desks on three distinct occasions across the six-week data collection timeline. In general, each group consistently drew upon a similar set of connections each time they



discussed the problem. For example, during Discussion 1, Group 1 repeatedly made connections between the table, the diagrams of hexagons, and the story as they completed the table in Question 1. During Discussions 2 and 3 Group 1 again referred to the table, the diagrams, and the story as they reasoned through which values to put in each line of the table. Likewise, Group 2 consistently used the connection to the numerical “add four” pattern in the table all three times they solved Question 1. While there were some changes in the groups’ responses across time, the relative consistency in the connections made by each group indicates that once a group makes connections among representations, these may remain consistent for a particular problem.

### Discussion

This study has illustrated the connections across representations made by two groups of students as they solved a non-routine generalization problem, explored the affordances of making different connections, and illustrated that each group’s initial connections remained fairly stable across six weeks.

### Implications for Research

These findings indicate that *making connections* may be a necessary, but not sufficient, characteristic for describing how students develop conceptual understanding in mathematics. Just as Wertsch (1998) noted that some number systems afford calculation by hand using standard algorithms, some connections may afford more mathematical insight than others, especially for developing generalizations. One connection, such as the recursive rule of “add four” in the table is common, but it is not necessarily the most useful connection for making generalizations about linear models (see also Kaput 1992 on the particular issue of recursive rules). Moreover, this study indicates that the connections that students initially made were relatively robust across time. For mathematics educators, this study invites a more systematic examination of which connections among which representations, and for which purposes or goals, have the most affordances for students’ mathematical reasoning.

Two possible follow up studies might investigate (a) what sequences of instructional activities promote productive shifts in the connections that students make while generalizing about linear functions, and (b) whether the affordances of different connections can be incorporated in assessments to better understand students’ emergent mathematical understandings.

### Connections to Practice

The data collection for this study coincided with a classroom unit on interpreting data, graphing, and reasoning with linear functions. While this analysis focused on the groups’ out of class discussions, the stability of the connections made by each group across time was surprising to both the researcher and teacher because, to us, the students’ in-class work appeared related to the goal of generalizing linear functions from data. For teachers and instructional designers, this study illustrates the well-known fact that student thinking can be oriented toward different goals than those intended by curriculum designers (Newman, Griffin, & Cole, 1989). Student thinking may also remain consistent from the learners’ own perspectives and thus appear to teachers as resistant to change through instruction. In this study, although the students did reason with linear functions, make sense of slope, and solve real life problems in the classroom, they did not seem to draw upon those classroom experiences while solving Hexagon Desks. While this study does not necessarily show how to help students draw on their classroom experiences, it does indicate that there is a continuing need to address this issue in both research and practice.

The difference in how each group made connections (and the affordances of those connections) is not suitable evidence to make generalizations about all bilingual students learning math in Spanish or English. First, these groups were chosen to illuminate two particular cases of students using social and linguistic tools, but not to represent of all bilingual students. Second, the data show that both groups were successful in different ways. For example, Group 1 was able to solve Question 2 by connecting the diagrams and story, but they were not able to solve Question 6. Conversely, Group 2 used a connection to the “add four” pattern in the table solve Question 6 but that did not work for solving Question 3.

## Conclusion

This analysis of the connections among representations made by two groups of students illustrates a critical issue faced by students as they navigate transitions along the learning continuum of school algebra. The connections that students make between representations are primary mediators of students' learning and understanding (National Council of Teachers of Mathematics, 2000; National Research Council, 2001). The findings of this study suggest that simply making connections across representations is not enough. Researchers and teachers need a better understanding of which connections students make, for which purposes, and how connections develop across time. An improved understanding of these issues can affect students' success navigating the continuum of school mathematics.

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