PRE-FRACTIONAL MIDDLE SCHOOL STUDENTS' ALGEBRAIC REASONING

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To understand relationships between students' quantitative reasoning with fractions and their algebraic reasoning, a clinical interview study was conducted with 18 middle and high school students. Six students with each of 3 different multiplicative concepts participated. This paper reports on the 6 students with the most basic multiplicative concept, who were also pre-fractional in that they had yet to construct the first genuine fraction scheme. These students' emerging iterating operations facilitated their algebraic activity, but the lack of a disembedding operation was a significant constraint in developing algebraic equations and expressions.

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Based in part on recommendations that improved fractional knowledge is critical for success in learning algebra (National Mathematics Advisory Panel [NMAP], 2008), researchers are starting to investigate how students' fractional knowledge is related to their algebraic reasoning (e.g., Hackenberg & Lee, 2011; Empson, Levi, & Carpenter, 2011). Since this research is in its infancy, there are numerous unexplored issues. One issue is how students who conceive of fractions primarily as parts *within* wholes may be challenged when working on algebra problems. These students' challenges may extend beyond the limitations of their fractional knowledge. However, little is known about how these students' fractional knowledge may assist or limit them in building basic algebraic ideas, such as making generalizations from quantitative relationships (Ellis, 2007; Kieran, 2007) and operating on unknowns (Hackenberg, 2010).

To understand relationships between students' fractional knowledge and their algebraic reasoning in the area of equation writing, a clinical interview study was conducted with 18 middle and high school students. Six students with each of three multiplicative concepts (Steffe, 1994) were invited to participate. These concepts have been found to significantly influence students' fractional knowledge (Hackenberg, 2010; Steffe & Olive, 2010), and they are based on how students produce and coordinate composite units (units of units).

The six students with the most basic multiplicative concept also conceived of fractions primarily as parts within wholes and had not yet constructed the first "genuine" fraction scheme, a *partitive fraction scheme* (Steffe, 2002, p. 305). So, these six students could be considered pre-fractional. That meant that the students did *not* conceive of a fraction like three-fifths as three one-fifths, related to but distinct from the whole. Instead, they thought of three-fifths as embedded within the whole—as five parts with three shaded. This view of fractions relies on being able to separate a quantity represented by a segment or rectangle into parts, a mental action we refer to as *partitioning*. However, it also relies on *not* being able to disembed a part from the whole while keeping the whole mentally intact, a mental action we refer to as *disembedding*. In general, pre-fractional students can learn to partition, but they do not yet disembed.

The purpose of this paper is to investigate relationships between the fractional knowledge and equation writing of the six pre-fractional students in the study. The research questions are:

1. How do pre-fractional students solve algebra problems that involve writing equations to represent relationships among unknowns?
2. How do pre-fractional students solve algebra problems that involve generalizing activity?
3. How are students' pre-fractional ways of operating related to their equation writing and generalizing activity?

A Quantitative and Operational Approach

Quantitative Reasoning

We conceive of students' quantitative reasoning as a basis for building fractional knowledge and algebraic reasoning (Smith & Thompson, 2008; Steffe & Olive, 2012). Approaching fractions as quantities means that we pose problems to students in which fractions are measureable extents, or lengths; these lengths may represent other quantities as well (e.g., weight). Approaching algebraic reasoning from a quantitative perspective means that unknowns are quantities for which a value is not known, but for which a value could be determined. So unknowns are potential values of quantities. In working with students we routinely ask them to make drawings of quantitative relationships, and we aim for students' fraction and algebraic notation to trace the quantitative reasoning in which students engage.

Operations, Schemes, and Concepts

Our work is also based on conceiving of mathematical thinking in terms of people's mental actions, or *operations* (Piaget, 1970; von Glasersfeld, 1995). Operations critical for fractional knowledge include *partitioning* and *disembedding* as mentioned above, as well as *iterating*, which is repeatedly instantiating a fractional part to make a larger fraction. Operations such as these are *interiorized* physical actions—that is, they arise from re-processing physical actions in such a way that they can be performed mentally, without having to be carried out materially.

Operations are the components of *schemes*, goal-directed ways of operating that consist of three parts: an assimilated situation, activity, and a result (von Glasersfeld, 1995). For example, if a student has constructed a partitive fraction scheme, then a situation of the scheme is a request to make a new length that is 3/5 of a foot. The activity of the scheme involves partitioning the foot into five equal parts, disembedding one of those parts, and iterating the part to make three such parts. The student then assesses the result of her activity in relation to her expectations.

For us, a *concept* is the result of a scheme that people have interiorized. For example, a student who has interiorized the result of her partitive fraction scheme can take that result, three-fifths consisting of three one-fifths, as a basis for carrying out more activity. This student could engage in problems such as determining what the result of partitioning each of the fifths into two equal parts would be, or how to re-make the whole if the given length is three-fifths of the whole.

Characteristics of Pre-Fractional Students

Students who are pre-fractional struggle in a variety of ways. For example, Olive and Vomvoridi (2006) have analyzed the case of Tim, who had not constructed a partitive fraction scheme by his sixth grade year. At that time, one feature of Tim's fraction scheme was that both a unit fraction and the whole referred to the same partitioned image: One-sixth meant a whole partitioned into six equal parts, and six-sixths meant the same partitioned whole. This idea about fractions led Tim to add up parts regardless of size. For example, in adding ½ and ¼, Tim said the answer would be 1/5 because ½ was one part and ¼ was four parts.

In short, pre-fractional students can engage in equal-partitioning of lengths (Biddlecomb, 2002; Steffe & Olive, 2010), but they cannot take a partitioned length as given prior to engaging in activity. For example, to share a 1-foot length of licorice fairly among five people, these students have to actually partition—they cannot imagine the partitioned length prior to making it. Research also shows that constructing a disembedding operation requires a significant reorganization of these students' ways of operating that can take as long as two years (Steffe & Cobb, 1988; Steffe & Olive, 2010).

Methods

Seven seventh grade students, 10 eighth grade students, and one tenth grade student participated in this clinical interview study. Participant selection occurred via classroom observations, consultation with students' teachers, and one-on-one, task-based selection interviews to assess students' multiplicative concepts. Six students with each multiplicative concept were invited to participate; this paper focuses on the six students with the most basic multiplicative concept. Three of these students were enrolled in a seventh grade mathematics class for struggling students; the other three students were taking an eighth grade pre-algebra class. The three seventh grade students and one of the eighth grade students received special education support for one period per day. All pre-fractional students had received some instruction in their mathematics classes on unknowns and equation solving.

Students participated in two 45-minute, semi-structured interviews, a fractions interview and an algebra interview. All students completed the fractions interview prior to the algebra interview, but the time between interviews varied from 3 weeks to 4 months. The interview protocols were refined in a prior pilot study (Hackenberg, 2009) and were designed so that the reasoning involved in the fractions interview was a foundation for solving problems in the algebra interview. For example, one fractions interview task was the following: "A 65-cm stack of CDs is 5 times the height of another stack. Can you make a drawing of the situation and determine the height of the other stack?" In the algebra interview, students were posed a similar situation but both heights were unknown. Students were asked to make a drawing and write equations to represent the situation. In addition, students completed a written fractions assessment (Norton & Wilkins, 2009) to triangulate claims about their fractional knowledge. This assessment confirmed that the students identified as pre-fractional were pre-fractional.

Each interview was video-recorded with two cameras, one focused on the interaction between the researcher and student, and one focused on the student's written work. The videos were mixed into one file for analysis, which occurred in three overlapping phases. The first phase of the analysis was to formulate a model (Steffe & Thompson, 2000) of each student's fraction operations, schemes, and concepts; equation writing and solving; and generalizing activity, to the extent possible over two interactions. Toward this end, the researchers viewed videofiles and took detailed analytic notes (Cobb & Gravemeijer, 2008), which included transcriptions, data summaries, memos, and conjectures. The resultant models provided the basis for responding to the first two research questions for this paper.

In the second phase of the analysis, the researchers looked across the students to articulate differences in how students with different multiplicative concepts solved the problems in each interview. Products of this phase included written syntheses of the ways of operating of students with a particular multiplicative concept, which provided an important backdrop for responding to the three research questions in this paper. Finally, in the third phase of analysis researchers examined how the operations, schemes, and concepts that constituted students' fractional knowledge were involved in students' equation writing and generalizing activity. This phase was the basis for responding to the third research question for this paper.

Analysis and Findings

Equation Writing and Multiplicative Relationships

Two of the six pre-fractional students, with significant coaching, wrote equations to represent multiplicative relationships between unknowns that were correct from the researchers' perspectives. Our analysis suggests that the students' emerging iterating operations were one reason these students were successful, but that the lack of a disembedding operation was a major source of the difficulties that even these two students experienced in conceiving of unknowns in multiplicative contexts. In this section we present one student's work on the first problem in the algebra interview to substantiate these claims.

The first problem in the algebra interview was the following:

> A1. *Cord Problem*. Stephen has a cord for his iPod that is some number of feet long. His cord is five times the length of Rebecca's cord. Could you draw a picture of this situation? Can you write an equation for this situation? Can you write another equation?

Initially all pre-fractional students made a drawing for A1 in which one of the lengths (represented by a segment or rectangle) was a little more than half of the other. Only two students refined their pictures to make a more accurate representation by iterating a shorter segment five times to make a longer segment. Only one of these two students, 7th grader Henry, wrote a multiplicative equation for A1 that was correct from the researchers' perspectives.

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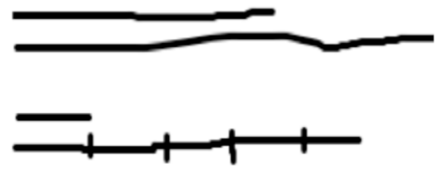

Figure 1: Henry's work on A1

In Henry's initial picture for A1, the segment representing Rebecca's cord length was longer than Stephen's, and Stephen's segment was a little more than half of Rebecca's (Figure 1, top two segments). Without any intervention from the interviewer, Henry reinitiated his activity and drew a small segment. Then he drew a copy of that segment below, and he proceeded to draw four more copies, pausing after each copy (but not lifting his pen). So he repeated one cord length five times, and this new segment represented the other cord length (Figure 1, lower two segments; hash marks have been added for clarity). Henry called the long segment Rebecca's and the short segment Stephen's. However, Henry switched these meanings when the interviewer restated the problem. Spontaneously initiating the repeating of a segment was novel, and it suggested that Henry had constructed an iterating operation that he would need for constructing more advanced fractional knowledge.

Henry's initial equation for A1 was "$S \cdot \mathrm{O} = R$cord," which he said meant "Stephen's cord times what Rebecca's cord is, equals Rebecca's cord." He said that he wrote an "O" to "leave it open," since he did not know the length of Rebecca's cord. Then, in discussion with the interviewer about how many of Rebecca's segments would fit into Stephen's in Henry's picture, Henry generated a correct equation, "$R \cdot 5$ = Stephens cord." In explanation, he changed the 5 to a 4, saying, "No, Rebecca's times four equals Stephen's cord, 'cause she already has one [of the segments]." This conflation suggests the lack of a disembedding operation: Rather than consider Rebecca's cord as one part of Stephen's, Henry appeared to think of Stephen's as five parts, one of which had to be Rebecca's, leaving Stephen with only four parts.

The researcher then posed a numerical example in order to test the equation: "Let's say Stephen's cord length is 15 feet; how long is Rebecca's cord?" Henry spent nearly 6 minutes determining Rebecca's cord length. He initially thought it would be 10 feet. Then he tried 5 feet and arrived at 15 feet. He appeared to be iterating an amount three times, because then he said "three, 9 feet." In this process, Henry extended the segment for Stephen's cord length by another segment the size of Rebecca's cord length, so Stephen's length then consisted of six segments (later Henry crossed off this part following questioning from the interviewer). We note that confusing "five times" and "five more than" is a sign of not having constructed iteration (Steffe & Olive, 2010, p. 182). So, despite Henry's later correction of his drawing, this work throws some doubt on whether Henry had indeed constructed an iterating operation for segments.

To explain his answer of 9 feet for Rebecca's length, Henry counted by threes along the first four parts of Stephen's segment, and then counted by ones ("13, 14, 15") along the fifth part. When the interviewer repeated back to Henry how he had counted along Stephen's segment, Henry changed his mind. "Yeah, hers is like three feet," he said.

Finally, the interviewer asked Henry about his equation—whether he wanted to use 4 or 5. Henry said four, although under questioning he agreed that 3 x 4 was not 15. Following that exchange, Henry changed his equation back to "$R \cdot 5$ = Stephens cord." When asked for any other equations he could write for the situation, he wrote "$5 \cdot 3 = 15$, $3 \cdot 5 = 15$, and $3 \div 15 = 5$."

Our current interpretation of Henry's work relies on the fractional operations we could attribute to him. Although the evidence is not incontrovertible regarding Henry's operation of iteration, Henry appeared to have something like iteration available—or becoming available—based on how he made his drawing for A1. The spontaneous change that he independently made in his drawing allowed him to create a quantitative foundation for his algebraic work that was a key reference during the rest of his activity. Since only one other pre-fractional student in the study made a similar drawing, we infer that creating this kind of drawing to show one segment and another that is five times longer is not a trivial achievement for a pre-fractional student.

In addition, although Henry received significant support from the interviewer in order to write a correct equation, we suggest that his emerging operation of iteration allowed him to make sense of the support that the interviewer offered in terms of questions about his picture. In contrast, none of the other pre-fractional students wrote a similar equation with similar questions—not even the pre-fractional student who generated a drawing similar to Henry's.

However, we suggest that Henry's lack of a disembedding operation, as shown most clearly in him changing the 5 to a 4 in his equation, was a constraint for him. That is, although with the support of the interviewer's questions Henry did return to a correct equation, it's not clear whether using 5 was a logical necessity for him. Indeed, without a disembedding operation it would be unlikely for Henry to make sense of a segment that is five times another, because that relationship appears to require thinking about the other segment as both embedded in and disembedded from the longer segment. So, without that operation, it would be more natural for Henry to think of the longer segment as "four more" than the original segment. This analysis indicates that writing equations representing multiplicative relationships between quantities would be quite challenging for students without disembedding operations.

Making Generalizations: Solving The Border Problem

In contrast to their work on A1, five pre-fractional students solved parts (a), (c), and (d) of the Border Problem, which has been used to introduce ideas of unknowns and variables to middle school students (Boaler & Humphreys, 2005):

A7. *Border Problem*. Below is a 10 by 10 grid with the squares on the border shaded.

(a) Without counting one-by-one, and without writing anything down, can you find a way to determine how many squares are on the border?
(b) Can you find another method?
(c) Can you apply your first method to a 6 by 6 grid?
(d) How would you describe in words how to use your first method on any grid?
(e) How would you use algebra to write an expression to communicate your first method to someone?

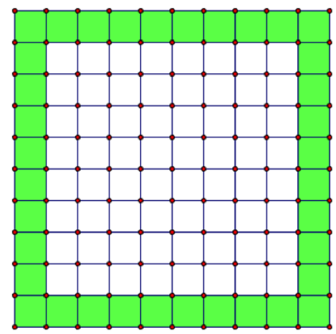

Figure 2: Border problem

All six pre-fractional students initially thought that there were 40 squares. Upon counting to check, five students adjusted their initial idea based on observations about counting the corner squares of the grid twice. Two students adjusted by subtracting 4 from 40. Three students, including Henry, adjusted by adding 10 and 10 for the top and bottom sides, and then adding 8 and 8 for the left and right sides (eliminating both corner squares from these sides). All five students applied their method to a new grid, a 6 by 6 grid, and verbally described their method to some degree. The two most detailed verbalizations were from Henry and another 7th grade student, Courtney, which we state below.

Due to time constraints, only these two students were asked to use algebra to communicate their methods (part (e)). Courtney said she did not know how to do that, even after discussion with the interviewer about using a letter to represent the number of unit squares in one row of the grid. However, Courtney did then apply her method correctly to a 15 by 15 grid without drawing that grid. Henry also had a discussion about part (e) with the interviewer, who suggested that x could represent the number of unit squares in one row. After asking Henry what x was in each of the first two grids (the 10 by 10 and 6 by 6), the interviewer asked if Henry could use x to write down an expression for the number of squares on the border. Henry wrote "x = top row 10" and then underneath "x = top row 6." Then he added the 10 and the 6 to get 16. So, no student made a correct solution to part (e) from the perspectives of the researchers.

Yet the five students who solved parts (a), (c), and (d) did generate a method for determining the number of squares on the border, used it on a grid of different size, and verbalized the patterns they observed. We assess that in doing so, they engaged in two forms of generalizing activity (Ellis, 2007): They extended their reasoning beyond the range in which it originated, and they began to identify commonalities across cases. However, we propose that the students' lack of a disembedding operation constrained the nature of their generalizations and prevented them from writing an algebraic expression. Although these conclusions were made from analysis of all data, we use Henry and Courtney as examples

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for explaining them, in part because these two students demonstrated some of the more advanced thinking of the pre-fractional students.

Henry's generalizing activity. In describing in words how to use his method on any size grid (part (d)), Henry said, "I'd tell them to do the top first, see how much in a row it would be [pointing at a row]. And then do the bottom, which is the same. And then after that, like, whatever number's at the end [corner], go to the next box [down] on the other side and put, like, put how much it is. Don't use the same number two times." When asked if by his last statement he meant "don't count a corner square again if you've already counted it," Henry agreed. The interviewer then asked how Henry knew, in the 6 by 6 grid, that the other side had to be four, and whether the four had any relationship to the six. Henry's response was inaudible. When the interviewer asked the same question about 8 and 10 in the 10 by 10 grid, Henry said "Huh?" and proceeded to label his drawing with numerals.

From this data excerpt, we conclude that Henry did not articulate the relationship between the 4 and the 6 and the 8 and the 10 structurally. In other words, he did not appear to see 4 as embedded in 6 and also separate from the 6 in terms of the side lengths of the grid (and similarly for 8 and 10). This means that in thinking about the grid he did not disembed 4 from 6 (or 2 from 6) while leaving the 6 intact—and we infer he did not do so because he had not constructed a disembedding operation. His comments do provide evidence that he knew two different numbers should be involved—that a person can't just add the same number four times as he initially did. But the lack of a disembedding operation contributed to Henry's generalization about adding the number of unit squares in the top and bottom rows, and then adding a different pair of numbers for the other sides of the grid. This generalization might lead to writing something like $x + x + y + y$ as an algebraic expression, but it would not lead to something like $x + x + x - 2 + x - 2$.

Courtney's generalizing activity. In contrast with Henry, Courtney subtracted 4 from 40 in solving the Border Problem. In writing down her method, she first wrote multiplication signs in between each of the four tens, changing them to addition signs under questioning from the interviewer. In verbally describing her method, she said, "Since a square has ten [sic] sides, on each one, I'd add ten plus ten four times and then I subtracted four 'cause I counted all four ends [corners], and I counted them twice. So I subtracted four since there are four sides. For the 10 by 10 I got 40 and then I subtracted 4 and I got 36." To clarify, the interviewer asked Courtney why she added 10 four times, and Courtney said it was because the square had four sides. At this point the interviewer did not probe for clarity about reasons for subtracting four. However, prior to this data excerpt and within it, Courtney said she subtracted four in order to not count corner unit squares twice. In fact, except for stating that she subtracted four due to there being four sides, her generalization does not seem problematic in any obvious way.

Yet based on our model of Courtney, we claim that she did not write an algebraic expression for her generalization in part because of how she thought about the "ten plus ten four times" and the 40. Since we knew Courtney had not constructed a disembedding operation, we knew that taking a number (such as 10) some number of times was a significant cognitive load for her (Steffe, 1994). When students like Courtney take a number (10) and repeat it, they do not consider these numbers (four 10s) as both embedded in and disembedded from the result (40). Instead, it's like the tens disappear after they have been used. In short, Courtney's method really was not $10 + 10 + 10 + 10 - 4$, structurally; we infer she did not generate awareness of the 10s as segments of the 40 in the process of and after computation. This conclusion is supported by Courtney's multiple ways of describing and notating the use of 10s to make 40. This analysis indicates that it was quite reasonable for Courtney *not* to know how to use the interviewer's suggestion to let x represent the number of squares in one row in writing an expression for her method, since the relationship between the number of unit squares on each side and the total number of border squares was rather ephemeral for her.

Discussions and Conclusions

This study contributes to understanding why pre-fractional students struggle with algebra. In particular, it suggests that these students' iterating operations may facilitate their representation of multiplicative relationships between quantities, and that these students' lack of a disembedding operation is a significant constraint in developing algebraic equations and expressions. It also suggests that students' fractional operations shape the generalizing activity in which they engage. For example, without a disembedding operation, pre-fractional students will be unlikely to distinguish amounts that are both contained within and separate from other amounts in a quantitative situation—and doing so is critical for creating a structural view of many situations that can be represented with algebraic notation.

Implications for algebra instruction for pre-fractional middle school students include the pressing need to develop curricular materials that provide support for helping these students advance their fractional and algebraic knowledge simultaneously. These materials need to be based on the ways and means of operating of the students so that these students will not be left out of making mathematical progress, and so that their mathematical thinking will not remain invisible or under-valued in an environment where extant curricular materials assume operations that these students are yet to construct.

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