TRANSITION FROM DERIVATIVE AT A POINT TO DERIVATIVE AS A FUNCTION

Jungeun Park
University of Delaware
jungeunpark124@gmail.com

Alfinio Flores
University of Delaware
alfinio@udel.edu

This paper explores how textbooks address two central concepts in differential calculus, derivative at a point and derivative function, make the transition from one concept to the other, and establish connections between them. We analyzed how the three most widely used calculus textbooks present these two aspects of the derivative, focusing on visual means and word use in the books. In contrast to their thorough discussion on the limit process for the derivative at a point, the books make a quick transition to the derivative function by "letting a point ***a*** *vary" and changing "f '(a) to f '(x)." Then, they graph f '(x) using several values of the derivative at a point. In addition, the books often use the term "derivative" without specifying which of the two concepts is meant, and are inconsistent in the use of letters, so that it is unclear whether a letter (a or x_1) denotes an arbitrary but fixed number or a variable.*

Keywords: Advanced Mathematical Thinking; Post-Secondary Education

Introduction

With a gradual growth in research in teaching and learning calculus, there have been several studies about students' thinking about the derivative. Most studies have reported students' conceptualizations about the derivative (e.g., Tall, 1987; Thompson, 1994), and their notations (e.g., Hahkioniemi, 2005; Zandieh, 2000) by addressing several mathematical aspects. This study focuses on two aspects: the derivative at a point as a specific value, and the derivative function as a function. Other researchers have emphasized these aspects (e.g., Oehrtman, Carlson & Thompson, 2008), but few studies have been done especially about the derivative as a function, nor about the transition and connection between derivative at a point, and the derivative function.

Motivation of this study came from Park's (2011) study about calculus instructors' and students' discourses on the derivative. The results showed that instructors addressed some aspects of the derivative implicitly in class using the word "derivative" without stating whether it was "derivative at a point" or "derivative function," and how these two concepts are related. During the interviews, students also used the word "derivative" without specifying and often to support incorrect notions such as "derivative as tangent line." From these results, we started wondering how to help students realize the relation and difference between derivative at a point and derivative function, make a transition from one to the other and build connections between them. As a first step, we decided to explore how widely-used calculus textbooks address the derivative as a point-specific concept and as a function. Specifically, we address the following questions:

1. How textbooks for Calculus I address the derivative at a point?
2. How textbooks for Calculus I address the derivative of a function?
3. Whether and how textbooks for Calculus I make a transition/connection between the derivative at a point and the derivative of a function?

This study is important for several reasons. First, it focuses on a central, but not yet sufficiently analyzed, relation between two main concepts of differential calculus, derivative at a point and derivative as a function. By studying students' opportunities to establish such relation through the material presented in the textbooks, if the analysis shows gaps or inadequacies in the presentations, we will be able to suggest ways instructors may complement how the books presented the idea. The textbooks analyzed in this study, which are used by over 70% college calculus instructors, share many similarities in their approaches to derivative. Second, exploring the relation between derivative at a point and derivative function is important because it offers calculus students an opportunity to revisit central aspects of function, namely a relation between thinking about function pointwise and across an interval. Though the concept of function is fundamental to understand calculus concepts, many students who received A's still have incomplete conceptions of function after their second calculus course (Oehrtman et al., 2008).

Theoretical Background

Function at a Point and Function on an Interval

There is a rich body of research on how students understand function, which also has provided several conceptualizations of functions. The studies, which address developmental stages of understanding functions, have made a clear distinction about function at a point and function on an interval (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Most studies describe the first stage of understanding function as being able to generate an output value of a function when an input value is given. A person at this stage would think of function as a value for a given input. Monk (1994) called this view of function "*pointwise understanding,"* and Dubinsky and McDonald (2002) called it "Action." The next stage is described as being able to see dynamics of a function. Monk (1994) called this stage "*across-time understanding,*" and described it as an ability to see the patterns in change of a function resulting from patterns in input variables. Dubinsky and McDonald (2002) called it "Process." Breidenbach et al. (1992) found that a transition from the first to the second stages is not natural, and some calculus students are at the first stage, and thus they have trouble seeing calculus concepts dynamically.

Derivative at a Point and Derivative as a Function

Existing studies on students' thinking about the derivative can be divided regarding the two views of functions. Studies about the derivative as a point-specific value showed that students' thinking about the limit is related to their thinking of local linearity (Hahkioniemi, 2005) and tangent line (Tall, 1987). Studies about the derivative as a function that mainly address co-variation showed the importance of what is varying in a function. Oehrtman et al. (2008) compared the rate of change of the volume of a sphere with respect to its radius (its surface area) and the rate of change of the volume of a cube with respect to its side (not its surface area). Thompson (1994) related the rate of change to students' thinking of the derivative.

However, few studies have been done about the relation between these two types of understanding of the derivative. Monk (1994) addressed these two types based on students' written answers on four survey problems, but did not give much detail about whether and how students related these two concepts. Park (2011) interviewed 12 calculus students and found that using one word "derivative" for both "the derivative function" and "the derivative at a point" was related to their conception of the derivative as a tangent line. The students were changing what the word "derivative" refers to in various contexts and used it as a mixed notion of a point-specific concept but a function, which the tangent line represents. They also used this idea to justify an incorrect statement, "a function increases if the derivative increases." Analysis of their class lessons about the derivative showed that the instructors were not explicitly addressing the derivative at a point as a number, and the derivative function as a function. In this current study, whether and how the calculus textbooks relate these two mathematical aspects will be explored.

Words and Visual Mediators

This study is based on the communicational approach to cognition (Sfard, 2008), which views mathematics as a discourse characterized by four features: word use, visual mediators, routines, and endorsed narratives. This study focuses on the first two features. A *word* in mathematical discourse can be used differently in a different context. For example, the word "derivative" is used as the derivative at a point and the derivative function (e.g., "is the derivative positive here?"). Quantifiers (e.g., one & any) play an important role to determine if "derivative" is a point-specific value or a function. *Visual mediators* refer to visual means of communication. This paper focuses on various notations of the derivative and letters for a point and variable. For example, if the derivative at a point is denoted as $f'(a)$, and the derivative function as $y = f'(x)$, "a" is used as a number, and "x" is used as a variable. The derivative at a point can be visually mediated by the slope of the tangent line and the derivative function by its graph.

Method

Based on Bressoud's (2011) study, we chose three textbooks that are widely used by Calculus I course instructors in the United States: one edition by Stewart (43%), Hughes-Hallett et al. (19%), Thomas et al. (9%). In each book, we explored the sections about the rate of change, and the derivative. We developed an analytical tool using an existing framework (Park, 2011). Though there were slight differences in each book, we identified five phases: (a) rate of change, (b) the derivative at a point, (c) transition, (d) the derivative function, and (e) connection. The first phase addresses the rate of change without using the word "derivative." The derivative at a point is defined in the second phase. In the third phase, the books make a transition to the derivative function, and define it in the fourth phase. Last, they connect back to the derivative at a point graphically. We examined book descriptions through their visual mediators and word use (Table 1). We focused on whether key terms—slope, rate of change, and derivative—were used as static or dynamic based on whether it is defined at a point, multiple points, or on intervals with a variable. Because books have limitations showing dynamics, we carefully looked at the descriptions for the figures including quantifiers and letters.

Table 1: Analysis Table

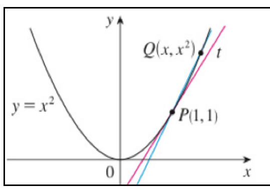
Stage	Visual Mediator			Word Use			
	Table	Graph	Symbolic Notations	Key Term	Static		Dynamic
					A point	Multiple points	

Results

In this section, we focus on the most-widely used book (Stewart, 2010) with the details of how we used the key words and visual mediators to reach our conclusions. The analysis of other books is addressed in the Discussion.

Velocity and Slope of Tangent

Stewart's (2010) *Calculus* addresses the slope and velocity in the chapter of Limit without using the word *derivative*. First, it shows how to obtain the slope of tangent line to a curve $y=x^2$ at $P(1, 1)$ using the point $Q(x, x^2)$ approaching P (Figure 1).



x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

$m_{PQ}=\frac{x^2-1}{x-1}$ …for the point $Q(1.5, 2.25)$… $m_{PQ}=\frac{2.25-1}{1.5-1}=2.5.$
The table…shows the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2.

Figure 1: Graph of $y = x^2$ and values of slope of secant lines (p. 45)

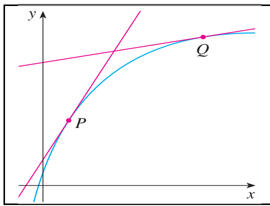
Using the same method, the book calculates the velocity of a ball after 5 seconds as *49 m/s* (the distance: $s(t) = 4.9t^2$), and relates its velocity at $t = a$ to the slope of tangent to the curve, $s(t)$. Here, the book addresses the slope of the tangent line to a curve and the velocity as point-specific concepts, at $x = 1$, $t = 5$, and $x = a$. The book used a as if a were a number rather than an arbitrary value or multiple values without stating that a could be any point. The dynamic aspect of the concepts was only addressed in the limit process finding the slope of tangent from secant lines. Thus, this section addressed the velocity and slope of a tangent line at a *specific* (single) point.

Derivative at a Point

The book calls the "special type of limit" in the slope and velocity "a *derivative*," and uses the word with phrases, "of a function," "at *a*," or an equation in this section. It rewrites the slope of the tangent line of $y = f(x)$ at $x = a$ as $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, and defines "the *derivative* of a function f at a point a" as the same limit (p. 107). The letter a is used only for a point until the book calculates "the derivative of a function $f(x) = x^2 - 8x + 9$ at the number a" as "$f'(a) = 2a - 8$" and finds the slope at (3, -6) as "$f'(3) = 2(3) - 8 = -2$." Though this calculation implies that 3 is one value of a, the book does not explicitly state it or write $a = 3$ (p. 107).

Rate of Change

The book defines the instantaneous rate of change of $y = f(x)$ with respect to x at $x = x_1$ as $\lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, interprets it as "the derivative $f'(x_1)$" and changes it to "the derivative $f'(a)$." It then gives two interpretations, "the slope of tangent line to a curve when $x = a$" and "the instantaneous rate of change…at $x = a$," and makes a connection between them (Figure 2, p. 108).



[On] the curve, $y = f(x)$…the instantaneous rate of change is the slope of the tangent to this curve at the point, where $x = a$. This means that when the derivative is large (and therefore the curve is steep, as at the point P), the y-values change rapidly. When the derivative is small, the curve is relatively flat (as at point Q). and the y-values change slowly.

Figure 2: Graphs of two tangent lines (p. 108)

For the cost of producing x yards of fabric, $C = f(x)$, the book explains "the derivative, $f'(x)$" as "the rate of change of the production cost with respect to the number of yards produced" in dollars/yard and asks to find or compare the meaning of $f'(1000) = 9$, $f'(50)$, and $f'(500)$ (p. 109).

In this section, the book uses the word "derivative" three times. "The derivative, $f'(x_1)$" indicates that it is defined at a "fixed point x_1" (p. 109). In Figure 3, "derivative" is used to describe the function behavior as in "when the derivative is large, the y value change rapidly" (p. 108). Because the book specified the point P, it is clear that the sentence is about the local function behavior near P, but it can be true anywhere on the interval if "the derivative" is used as a function. At the end, the book calls all rates of change of various functions at several points "derivatives." It uses the notation, $f'(x)$ for the first time. In the fabric problem, it interprets $f'(x)$ as if it were a point-specific value, but gives its units in general terms using the units of different quotients without making a connection to its interpretation. In the second problem, it interprets "$f'(1000) = 9$," as "when $x = 1000$, C is increasing 9 times as fast as x." Though $f'(1000)$ was used as a value of $f'(x)$ at $x = 1000$, the relation between notations, $f'(1000)$ and $f'(x)$, was not stated.

Transition from the Derivative at a Point to the Derivative of a Function

The book summarizes that all previous discussions were about "a fixed point," which confirms the word, "derivative," "x" in the fabric example and "a," in graphs as point-specific values. Then, in $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, the book changes the "point of view and let[s] the number a vary, … replace[s] a by a variable x, and…obtain[s] $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$" (p. 114). Here, the nature of "a" was specified as "vary[ing]" and connected to the "variable, x."

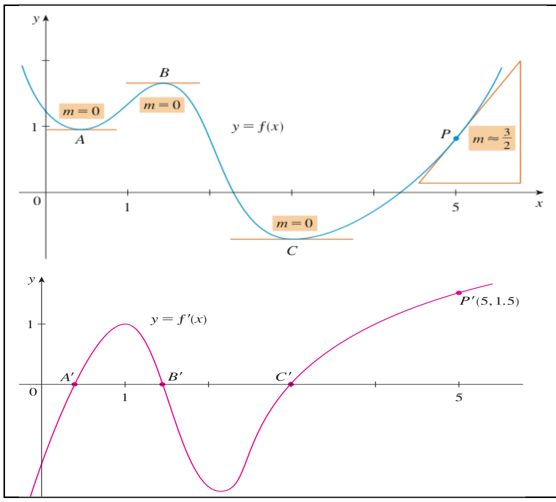
The Derivative Function

The book defines "$f'(x)$ as a new function" that assigns to "any number x…the number $f'(x)$," and connects it to "the slope of the tangent line to the graph of f at the point $(x, f(x))$" (p. 114). It also emphasizes that the variable x in $f(x)$ and $f'(x)$ are the same by comparing the domain of f', $\{x | f'(x)$ exists$\}$ that may be smaller than the domain of f (p. 114).

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

Connection from the Derivative Function to the Derivative at a Point

The book then graphs *f '(x)* using slopes of tangent lines to the curve, *f(x)* (Figure 3).



We can estimate the value of the derivative at any value of *x* by drawing the tangent at the point (*x*, *f(x)*) and estimating its slope. For instance, for $x = 5$ we draw the tangent at *P* in [the] Figure and estimate its slope to be about 3/2 , so $f'(5) \approx 1.5$. This allows us to plot the point *P'* (5, 1.5) on the graph of *f '* directly beneath *P*. Repeating this procedure at several points, we get the graph shown in Figure 2(b). Notice that the tangents at *A*, *B*, and *C* are horizontal, so the derivative is 0 there and the graph of *f '* crosses the *x*-axis at the points *A'*, *B'*, and *C'*, directly beneath *A*, *B*, and *C*. Between *A* and *B* the tangents have positive slope, so *f '*(*x*) is positive there.

Figure 3: Graphs of a function and its derivative function (p. 115)

In Figure 3, the book makes a connection from the derivative function to the derivative at a point by stating the value of the derivative at "any" point of *x* using the slope at the point, finding the slope 1.5 at $x = 5$, and plotting (5, 1.5) for *f '(x)*. It again uses the point-wise approach to find the zeros for "the derivative." Then, it uses the interval-wise approach to determine whether *f '(x)* was positive or negative between these zeros. Here, the word, "derivative" first is used as *the derivative function* because it was defined "at any value." The second one in "the derivative is zero there and the graph of *f '* crosses the *x*-axis" is used as a point-specific value. To refer to the function that the second graph represents, the book consistently used the notation *f '(x)*. When it describes the sign of the "slope" of "tangents" on intervals, it used the singular "slope" instead of "slopes." Though "the slope" can be inferred as "the slope" as a function because the book was using "the slope" for several values, it would have been "the slopes of the tangents."

Summary

To address the concept of the derivative, Stewart (2010) (a) uses the velocity and slope at a point, (b) defines the derivative of a function at a point, (c) interprets it as the instantaneous rate of change, (d) makes a transition by letting point *a* vary and replacing it with variable *x*, (e) defines the derivative of a function, and (f) constructs the graph of *f '(x)* using the slope of tangents to $y = f(x)$. Table 2 shows key words and visual mediators used in each of these phases.

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

Table 3: Visual Mediator and Word Use in Stewart (2010)

Phase	Visual Mediator			Word Use			
	Table	Graph	Symbolic Notations	Key Terms	Static		Dynamic
					A point	Multiple points	
(a)	Fig. 1	Fig. 1	$\frac{x^2-1}{x-1}$	The slope	At the point (1, 1)		Limit. “Values close to 1”
			49m/s	The velocity	After 5 seconds		
			$\frac{4.9(a+h)^2-4.9a^2}{(a+h)-a}$	The velocity The slope	At time $t = a$ At point P		Limit. “As h approaches 0”
(b)			$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$	The derivative of a function f	At a point a		
			$f'(a) = 2a\text{-}8$	The derivative of $f(x) = x^2\text{-}8x\text{+}9$	At the number a		
			$f'(3) = 2(3)\text{-}8\text{=-}2$	The slope of the tangent line	At (3, -6)		
			$\lim_{x_2\to x_1}\frac{f(x_2)-f(x_1)}{x_2-x_1}$	Rate of change The derivative $f'(x_1)$	At $x = x_1$		
(c)		Fig. 2	$f'(a)$	Derivative		At the points P & Q	
			$f'(x)$	Derivative Rage of change of C	With respect to x		
			$f'(1000) = 9,$	The rate of increase	After 1000 yd		
			$f'(50), f'(500)$	The rate of increase		After 50 & 500 yd	
(d)			$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$	Derivative	At a fixed point a		Let a vary
(e)			$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$	Derivative			[of] variable x
(f)			$f'(x)$	A new function			Any number x
		Fig. 3	f'	Derivative Graph of f'	At $x = 5$	Where f' crosses the x axis	Any value of x

As shown in Table 2, Stewart (2010) develops the concept of the derivative from a number to a function by showing graphical representations of the slope of a single tangent line, the slopes of multiple tangent lines, and the derivative of a function. It also uses the slope, the rate of change, the derivative at a specific point, multiple points, and any points by using numbers and letters with or without subscripts and changing what those key words represent. Graphical notations were consistent with such development: first a single tangent line, then multiple tangent lines, and graph of $f'(x)$ in turn (see Figures 1, 2, & 3). However, other visual mediators, letters for a point and variable were not consistent. Though the book mainly used *a* for a single point or multiple (discrete) points, and *x* for the variable, in the second and third examples in phase (b), "*a*" was used as if it were a variable, because *3* is substituted in *a* in the next step. The book does not mention that *a* can be any value. In a later section, it calls "*a*" "the fixed point." Then, in phase (c), it again uses "*a*" for multiple points, which seems to transfer what "*a*" represents from a single value to any values. However, in the next step, it uses "$f'(x)$" as if it were one value of the rate of change of a cost function and interprets $f'(1000)$, $f'(50)$, and $f'(500)$ without making a connection back to $f'(x)$ or mentioning they are the specific values of $f'(x)$.

The word "derivative" is also used inconsistently. First, it is used as *"the derivative* $f'(x_1)$*"* as the rate of change at a point, and again in "the derivative $f'(x)$ with respect to *x*" as if it were a point-specific concept (before the book defines the derivative function, $f'(x)$). Then, in Figure 3, the book uses the word "derivative" twice: one for the derivative of a function (at any points), and the other one for the derivative at points where f' crosses the *x* axis. The word is used without its referent–the derivative function or the derivative at a point–or notation–$f'(x)$ or $f'(a)$. The book relates these two concepts twice. First, it makes a transition from the derivative at a point to the derivative function by letting "*a*" vary and changing "*a*" to "*x*." Second, after defining the derivative of a function, it makes a connection back to the derivative at a point based on the slopes of several tangent lines to the original function at discrete points.

Discussions and Conclusions

As mentioned earlier, the textbooks address the concept of the derivative first as the velocity and slope without using the word "derivative," define the derivative of a function at a point, and then the derivative function. With slight differences in representations, the books we analyzed have some common characteristics in connecting the derivative at a point and the derivative of a function. First, the use of numbers and letters with or without subscripts is not consistent. For example, Thomas et al. (2010) uses $t = 1$ and $t = 3$ in a problem statement and $t_0 = 1$ and $t_0 = 3$ in its solution. It also uses a letter with a subscript x_2 for a value approaching a fixed value x_1. Second, the word "derivative" is not used explicitly; most times, it is used without its referent, the derivative at a point or the derivative function. Especially, when the word is used after defining both concepts, it is not clear whether "derivative" is used as a point-specific value or as a function. With this implicit use of the letters and key words, the derivative at a point as a value of the derivative function is also not consistently addressed. For example, all three books use notations $f'(x)$ and $f'(a)$, and substitute a number in *x* or *a* before mentioning that the concept of the slope or the rate of change can be considered at more than one (or any) point on an interval or defining the derivative function. To define the derivative function, they all change the view to let "*a*" or "x_0", which used to be a fixed value, "vary" and change it to "*x*." After the definition, they show the graphing process of the derivative function based on the slopes to the curve $y = f(x)$. In this process, the word "derivative" is also used implicitly, which is problematic because they are graphing "the derivative function" based on "the derivative" at discrete points. Hughes-Hallet et al. (2010) even draws the graphs of a function and its derivative function on one *x-y* plane, which does not show that they represent different values, such as distance and velocity.

Calculus I is a first college course, in which students practice abstract mathematical thinking and prepare for upper level mathematics courses. Mathematicians, including textbook authors, may think that students have mastered the concept of function before they start the course. However, many studies show that this is not necessarily true; calculus students do not always have complete understanding of function in secondary level, and thus have trouble seeing the derivative as a function (Park, 2011). Calculus books cannot and need not include all the explanations of a function, which should be addressed in the previous

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

mathematics classes. However, inconsistent use of key words and visual notations supporting the concept of the derivative as a point-specific value and as a function may confuse calculus students who do not have a solid understanding of a function. The way the concepts of the derivative are built—(a) heavy discussion on the limit process in the derivative, obtaining the slope of the tangent from a sequence of secant lines; (b) a simple transition from the derivative at a, $f'(a)$, to the derivative function $f'(x)$; and (c) graphing $f'(x)$ based on several values of $f'(a)$– is not consistent with the way the concept of function was built before. Changing a view of seeing "*a*" as a fixed value to any values may not be *simple* to students and graphing $f'(x)$ after giving its definition may not be ideal. Constructing the derivative function based on the derivative at discrete points before defining the derivative function may remind students about how a function was constructed and thus help them guess what those values represent and how they change as x values change, and finally think about the derivative as a function before they see the formal definition.

References

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247–285.

Bressoud, D. M. (2011, June). The calculus I instructor. *Launchings*. Retrieved from http://www.maa.org/columns/launchings/launchings_06_11.html

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton et al. (Eds.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 273–280). Dordrecht, Netherlands: Kluwer Academic.

Hahkioniemi, M. (2005). Is there a limit in the derivative? Exploring students' understanding of the limit of the difference quotient. *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (Vol. 14, pp. 1758–1767). Sant Feliu de Guíxols, Spain.

Hughes-Hallet, D., Gleason, A. M., & McCallum, W. G. (2010) *Calculus: Single and multivariable*. *New York:* Wiley.

Monk, G. S. (1994). Students' understanding of functions in calculus courses. *Humanistic Mathematics Network Journal*, *9*, 21–27.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27–41). Washington, DC: Mathematical Association of America.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, *14*, 235–250.

Park, J. (2011) *Calculus instructors' and students' discourses on the derivative*. Unpublished doctoral dissertation, Michigan State University, East Lansing.

Sfard, A. (2008). *Thinking as communication*. Cambridge, England: Cambridge University Press

Stewart, J. (2010) *Calculus*. Mason, OH: Brooks/Cole Cengage Learning.

Tall, D. O. (1987). Constructing the concept image of a tangent. *Proceedings of Eleventh International Conference of Psychology of Mathematics Education,* (Vol. 3, pp. 69–75). Montreal, Canada.

Thomas, G. B, Weir, M. D., Hass, J., & Giordano, R. F. (2010) *Thomas' calculus: Early transcendentals*. Boston: Pearson Addison-Wesley.

Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2/3), 229–274.

Zandieh, M. (2000). A theoretical framework for analyzing students understanding of the concept of derivative. CBMS *Issues in Mathematics Education*, *8*, 103–127.