FINE-GRAINED ANALYSIS OF TEACHER KNOWLEDGE: PROPORTION AND GEOMETRY

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In this study, we considered how four teachers reasoned about proportional situations involving geometry. We used pilot interviews conducted with in-service middle grades teachers around a set of proportional tasks. Analysis relied on a knowledge in pieces lens to uncover the resources that each teacher used in solving the tasks and which resources seemed to support the reasoning. Our findings suggest that this is an area of proportional reasoning in which multiplicative comparisons are important, but insufficient, for supporting reasoning. We also found that these teachers did not invoke the constant of proportionality in ways that supported their reasoning.

Keywords: Teacher Knowledge, Middle School Education. Mathematical Knowledge for Teaching

Purpose of the Study

Teacher knowledge matters. This is a conclusion reached by scholars and researchers alike (e.g., National Mathematics Advisory Panel, 2008; Hill, Rowan, & Ball, 2005). However, the nature of that knowledge has been less thoroughly studied. To date, three main approaches have dominated the landscape of teacher knowledge research. The most enduring approaches have involved using proxy measures to seek correlations between teachers' knowledge and particular aspects of their backgrounds such as courses taken in their preparation programs (e.g., Begles, 1975; Goe & Stickler, 2008). These studies have yielded inconclusive results about the relationship of teacher knowledge to student achievement. A second approach has been the use of assessments that attempt to measure the amount of knowledge teachers have (e.g., Baumert et al., 2010; Hill, Rowan, & Ball, 2005; Post, Harel, Behr, & Lesh, 1988). These studies have uncovered a host of gaps in teacher knowledge. A third approach to studying teacher knowledge, exemplified by Ma (1999), relies on qualitative approaches to understand teacher thinking. While some of these studies have focused on teacher deficiencies as the basis for developing theories of teacher knowledge (e.g., Ma, 1999), others have used the fine-grained analysis of teacher understanding to uncover important aspects of teacher reasoning with an eye toward supporting further teacher development (e.g., Izsák, 2008; Orrill & Brown, 2012). The current study continues this third tradition by carefully considering teachers' reasoning about particular proportional situations. Here, we considered how the teachers' knowledge about proportions and geometry would work across the three situations.

Theoretical Framework

Consistent with this kind of approach to the study of teacher knowledge, we relied on the knowledge in pieces theory (e.g., diSessa, 1988, 2006; Smith, diSessa, & Roschelle, 1993) to guide our analysis of teachers' reasoning. Knowledge in pieces is appropriate and useful because it posits that our understandings are stored as fine-grained knowledge resources invoked in a given situation. In the knowledge in pieces view, as in other research focused on knowledge organization (c.f., Bédard & Chi, 1992), the development of expertise (i.e., learning) involves building connections and refinements that allow appropriate pieces of knowledge to be invoked in various situations. Thus, knowledge in pieces allows us a lens for considering *coherence* of knowledge. For those with more coherent knowledge, more knowledge resources should be readily invoked whereas knowledge resources for someone with less coherent knowledge would like be more haphazardly invoked. Knowledge in pieces allows us to consider that people may have understandings that are tacit in a given situation rather than assuming that unseen knowledge is missing. Further, it allows us to consider that any individual's knowledge is reasonable in isolation–that is, people are making sense of their world in ways that are reasonable to them given the knowledge they have invoked for the situation. This view is removed from the knowledge deficit models that choose to focus on the mathematics teachers may not know. It is also important because it allows us to begin to operationalize the notion of *coherence* which appears often in scholarly pieces about desired teacher knowledge (e.g., Ma, 1999; NMAP, 2008; Silverman & Thompson, 2008) but that has been rarely operationalized.

Methods

Participants, Context, & Data Sources

This study was an exploratory, qualitative study conducted as a pilot effort for a larger project. The participants were four middle grades teachers: one fifth-grade teacher (Aubry – all names are pseudonyms), one seventh-grade teacher (Liam), one eighth-grade teacher (Autumn), and one elementary mathematics coach who had also been a middle school teacher (Luke).

The data collected consisted of a set of clinical interviews (Ginsburg, 1997) focused on three key areas of proportional reasoning: geometric applications of proportional reasoning, the relationship of fractions and proportions, and the appropriateness of using proportional reasoning to solve a given task. All interviews were conducted by the first author with occasional questions from the second author. All of the interviews were videorecorded with two cameras, providing a record of all written work and interactions with the technology as the participants worked on each task. Participants were told that the purpose of the pilot study was to "try out" an evolving set of tasks about proportions that were being piloted for the larger study. The participants were interviewed for 60-90 minutes at a time and each participated in two or three separate interviews that spanned proportional reasoning more broadly.

Data Analysis

All data were analyzed by mapping the ideas the participants used in solving each task (e.g., Empson, Greenstein, Maldonado, & Roschelle, 2013). Then, we distilled these maps by looking for the key proportional ideas that emerged as the participants reasoned about the task. We then, created a timeline of the key events in solving the task. Timelines included key mathematical phrases from the interview with the prompts placed inline so we could determine the extent to which an idea was naturally emerging or being prompted by the interviewer. We then analyzed the solution path to determine which mathematical resources they had invoked and whether those ideas were supporting a reasonable solution of each task. To allow ourselves to understand the participants' approaches more broadly, we transferred this information into a cross-case table that allowed us to evaluate each task to see what knowledge resources were invoked and whether those resources seemed useful to the participant(s) invoking them.

The Tasks

Each task is briefly described below including a description of the task and our reasons for including it in the protocol. The first two tasks were intentionally developed so that neither was a comparison or missing value problem (Lamon, 2007) as our intentions were to understand how teachers reason about proportional situations rather than understanding whether they can determine correct responses to procedural items. Tasks were modified slightly from one participant to the next in response to our desire to refine the items, however the essence of the questions was the same among the four participants.

The Thermometer Task is a Geometer's Sketchpad (GSP) sketch that had two parts. The first showed two lines that related through a linear relationship, thus as the user drags the slider, they increase in length at the same rate. The second set of thermometers showed two lines that were proportional to each other, thus the second is always 3/5 the length of the first (Figure 1). This item was developed to determine the extent to which teachers recognize proportional and linear situations presented visually.

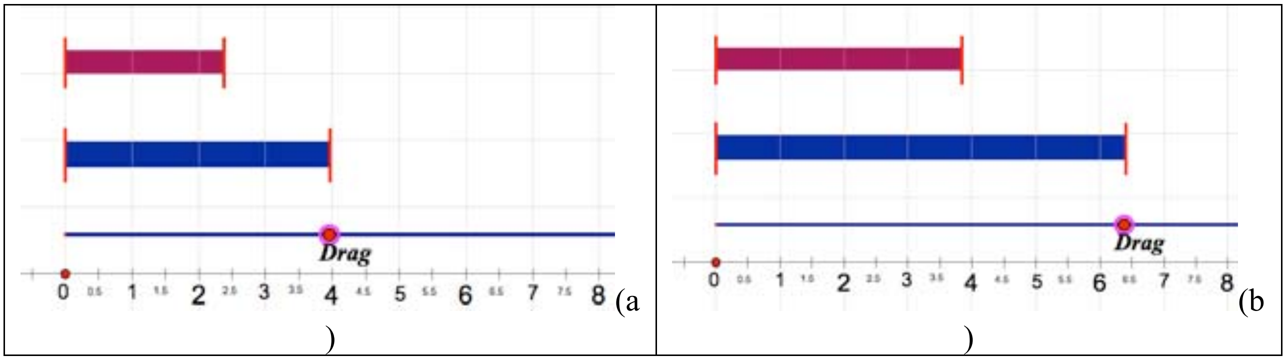

Figure 1: Proportional Thermometer Task (a) Initially and (b) After Dragging the Point.

The Smiling Guy Task was another GSP sketch that offered a fixed picture of "Smiling Guy" and a version of him that scaled up/down in a proportional way as the slider was moved (Figure 2). The item was based on a task about an imaginary character named "Wimpy" who liked to visit the hall of mirrors (Ben-Chaim, Keret, & Ilany, 2012). Based on early responses to the Santa task and to the original Wimpy task, we determined that our participants may not be attending to area as they reason about the changing size. We hypothesized that the dynamic nature of this new task would support more area-focused reasoning. Luke was not interviewed for this task.

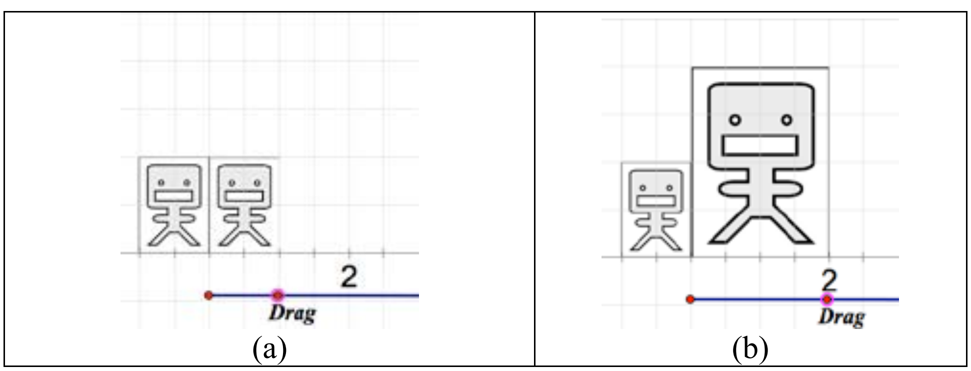

Figure 2: Smiling Guy Task (a) at Starting Point and (b) After Dragging

The Santa Task was adapted from De Bock, Van Dooren, Janssens, and Verschaffel (2002). In their study, high school students were given the task along with a set of prompts to determine

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the persistence of linearity in students' thinking. Because their results about the persistence of linearity were compelling, we wondered whether middle grades teachers would perform similarly. In this paper-based task Santa has been painted on a bakery door and the task asks how much paint will be needed if the painter were to paint the same picture three times taller on a grocery store window. A series of scaffolding situations, some from DeBock et al. (2002) and some developed for our participants, were provided to support area reasoning in determining the correct paint amount.

Bart is publicity painter. In the last few days, he had to paint Christmas decorations on several store windows. Yesterday, he made a drawing of a 56 cm high Santa on the door of a bakery. He needed 6ml paint. Now he is asked to make an enlarged version of the same drawing on a supermarket window. This copy should be 168 cm high. How much paint will Bart approximately need to do this?

Figure 3: Initial Presentation of the Santa Task.

Results

In this section, we briefly highlight observations from each task. Our goal was to understand how the teachers leveraged their knowledge resources to address each of the tasks. While each participant used a unique approach to each task, we present the findings here by looking for the particularly notable commonalities and differences between the teachers.

Thermometers Task

In the Thermometers task, we noticed a split in our participant pool. Autumn and Luke both relied on multiplicative reasoning—comparing one thermometer to the other—and making and testing conjectures to develop reasonable responses to the task. Both were ultimately able to identify that the first relationship was linear and the second was proportional. They relied on equivalent ratios to make this determination. For example, Autumn described the first (linear) situation saying, "I mean, the gap is always two until the initial start. Oh, the initial start, so zero to two. So there's always just a two lead gap. So as far as proportions go, no, that would not be good." Clearly showing that one of her criteria for differentiating proportions was that the "gap" between bars should not always be the same. This idea was reinforced on the second thermometer when she immediately tested the bars to see whether each could be 0 at the same time. Yet, when she declared that the second pair of bars was proportional, she had already written down some equivalent ratios as she shared, "if these two are proportional, then the difference when you increase the temperatures would have to be proportionally the same…So if this increases by 1, this would have to increase by 3/5. If this increases by 5, this would have to increase by 3." She went on to explain that the length of one bar could be found relative to the other bar (e.g., the red bar is 3/5 of the blue bar.) To us, this demonstrated that she was invoking a number of knowledge resources including equivalence, multiplicative relationships, and a definition of proportion that included the (0,0) instance.

Like Luke and Autumn, Liam and Aubry focused on the difference between the length of the blue line and the length of the red line to come to different conclusions. Liam determined that there were not proportional "because they're not the same". This was not an idea well elaborated as Liam moved on to describe how he would find one specific instance of the relationships between the bars. Aubry, on the other hand, felt they were proportional because "it's constant. It doesn't matter—if I went over—like if I went to 12, this would be 10. If I went to 10, this would be 8." For Aubry, the idea of constant, as it related to proportionality, seemed to refer not to a particular multiplicative constant relationship between the bars, but rather to any constant relationship.

However, for the proportional thermometers, they referred to a constant change focused on how much length was added with each move of the slider. Both participants maintained that there were not generalizable relationships between the thermometers on the proportional version. In fact, Liam pondered, "If red is 3/5 of blue, how can I talk about that as a proportional relationship?" The participants' responses indicated problematic use of knowledge resources related to the idea of constant in a proportion. Liam's response suggested that constant might mean "the same", though it was hard to know whether he meant the same length or the same rate of increase. However, his statement of not knowing how to think about red as 3/5 of blue in a proportional way was revealing in terms of highlighting that multiplicative reasoning was not being invoked by this task for him.

Smiling Guy Task

In the Smiling Guy Task, Autumn again used multiplicative language to compare the new Smiling Guy to the original. However, she rejected that the new Smiling Guy was proportional to the original because the ratio of areas changed as Smiling Guy increased or decreased in size. Thus, we assert that Autumn understood that there should be a comparison that yielded equivalent ratios, but was unable to identify the correct comparison between images. In contrast, Liam and Aubry accepted that Smiling Guy was proportional, but both again relied on reasoning that sounded additive. They explained that for every two units wider Smiling Guy gets, he also gets two units taller. We noted that this seemed almost like a formula—whatever you do to the top, you have to do to the bottom—rather than reasoning about the ratio as a coordination of two units (Lobato & Ellis, 2010). Thus, multiplicative reasoning used by Autumn did not yield a correct answer because she the understanding of constant she invoked was not adequate, whereas procedural reasoning from Liam and Aubry did yield correct answers.

When prompted about the relative size of the new Smiling Guy to the original, Autumn easily replied using multiplicative reasoning that took area into account. In contrast, Liam and Aubry struggled to name the relative size of the new Smiling Guy. They tried to make sense of the new size by relying on statements that included the phrase "for every" to explain the change. For example, when estimating how much larger the new Smiling Guy was than the original, Aubry said, "For every two units (on the slider scale) larger, it's two times as big." This further suggested that she was not coordinating units, but instead relied on application of a rule that says whatever is done to the width needs to be done to the height. It also demonstrated the problems the participants had with area. As shown in Figure 2, when the slider is pulled "over 2", the resulting Smiling Guy is four times larger than the original, but he is two units taller.

Santa Task

In the Santa Task, we again saw Autumn and Luke using multiplicative comparisons to talk about the relative size of the new Santa to the original. Consistent with the findings reported for high school students in De Bock et al. (2007), however, these participants initially attended only to the height of the new Santa rather than the area of the Santa for reasoning about the amount of paint needed. In contrast, Liam and Aubry capitalized on the appearance that this task was a missing-value problem (Lamon, 2007) by changing from their previous additive approaches. Instead, both set up proportions and used within space reasoning to find the scale factor between the two ratios (e.g., 168 is 3 times more than 56, so they multiplied the 6 ml by 3 to find 18ml of paint needed). As with Luke and Autumn, they only attended to height.

Interestingly, Luke, Aubry, and Liam rejected the use of area throughout the scaffolds included in the task. One such scaffold presented the Santa images with boxes drawn around them and asked whether this was a strategy that might be helpful for a student to use. All of the participants were readily able to see that the smaller image tiled the larger image, yet they continued to reject area reasoning for this task. Luke explained his rejection of this strategy saying, “"Like in the original problem, they didn't mention anything about width or anything like that. It was just strictly based on height. So I was just seeing it just in that relationship - a ratio. Here, they're literally seeing how many of these copies will cover this larger copy - will fill up that larger copy.” Liam and Aubrey made similar statements. This suggested to us that the participants had knowledge resources available to them (e.g., tiling the small Santa on the larger one to determine relative size correctly) that they were unable to capitalize on in reasoning about the task.

In contrast, Autumn began to question her assertion of only attending to height when presented with the first scaffold, which presented two different solutions found by students in a fictitious class (18 ml and 54 ml). She replied,

> [The students] were looking at the… oh, just the height. Oh I could even be wrong. I mean we’re not considering maybe the whole area… I don’t know. I mean I really don’t know necessarily, I might be over thinking it, but how would they get… I’m thinking that these students thought of it in terms of proportions for something to be this high if it has to fill up that much paint, that to do this much 168, you need a proportional amount of paint and for the other students I’m honestly not sure.

By the end of the task, which included three additional prompts, including the one in which boxes are drawn around the images, Autumn concluded that area was the necessary measure rather than height. Based on our analysis of her reasoning, we believe that she realized perimeter (another option) would only allow her to know how much paint would go around Santa and that queued her to invoke her area reasoning.

Discussion

From this small-scale study, it became clear that the ability of teachers to invoke multiplicative reasoning varied, and when invoked it supported better reasoning about the situations than other (e.g., additive) reasoning. However, multiplicative reasoning was not enough to consistently yield correct reasoning in these tasks. Other knowledge resources seemed important to combine with multiplicative reasoning. In our analysis, we noted two such resources: understanding what is constant in a proportion and understanding the role of equivalence in proportions. We also noted that procedural applications of proportional problem solving came with trade-offs of their own.

Understanding the constant in a proportion seemed potentially important, but elusive for these participants. For example, Autumn provided reasoning about the Smiling Guy task that compared the original to the new Smiling Guy in reasonable ways given that she did not exhibit an understanding of the appropriate constant in the relationship. She knew that the ratios needed to be constant (i.e., equivalent), but was looking at the wrong ratios. In contrast, Liam responded, when asked specifically about the constant of proportionality, that he did not know what that was. For the four teachers in this pilot effort, the constant of proportionality was not invoked in ways that allowed us to understand how the teachers might understand it as a concept or use it in their problem solving. There is a need for careful consideration of what teachers need to know about the constancy of proportions and how to support them in developing that understanding.

The second particularly helpful knowledge resource we noted what the use of equivalence. In these tasks, Luke and Autumn both regularly checked for equivalence as they worked. They invoked a definition of proportionality that allowed them to rely on such equivalence. Luke was very systematic about this, in that he created a t-chart for each situation and used that to reason both across and down the chart to ensure equivalence before declaring any situation proportional.

As with prior research (e.g., Karplus, Pulos, & Stage, 1983; Singh, 2010) we found that existing algorithms obscured our ability to fully understand the teachers' sensemaking in places. This was particularly true in the Santa Task in which Liam and Aubry relied on a particular algorithm focused on one particular relationship to approach the task. We ponder whether this was an invocation of the common missing-value approach. We also ponder whether they understood the mathematical similarity of Santa to Smiling Guy. This raises questions about how teachers think of an equivalence class of relationships (e.g., Lobato & Ellis, 2010) if they treat a single instance in a different manner than a dynamic relationship. The use of the apparent missing-value approach also raised questions about whether Liam and Aubry were reasoning about ratios as relationships involving coordination of units or whether they were applying a rule of 'do to the top what you do to the bottom.'

Combined, we assert that these data suggest that the teachers had understandings that were not invoked (e.g., area) as well as those that may have been underdeveloped (e.g., the constant of proportionality). However, the teachers did bring a number of resources to their analysis of the tasks. This suggests the needs for further research into this area so that we are better able to support teacher development of proportional reasoning. From this small study, we hypothesize that without perturbation, our participants did not face their weaknesses in understanding, even when scaffolds were provided to perturb their thinking, because their schemes were not challenged. Thus, being able to solve missing-value problems as in Santa may have demonstrated their ability to apply a procedural understanding in absence of conceptual grounding. This may limit the utility of the approach for a wider range of proportional tasks. Because these are preliminary data, this is a question for further exploration.

Finally, we see from these data that teachers do, in fact, have knowledge that may not be invoked in particular situations. This is important because it suggests that all of their knowledge resources may not be equally available to students in the classroom. It is also important because it suggests that more research is needed in this area. It is clear from this study that some of the knowledge not invoked was, in fact, available to the teachers more broadly. For example, they all demonstrated an easy ability to think about the Santa task using a tiling approach that allowed them to talk about area, yet they chose not to use that knowledge in thinking about the amount of paint needed for the task. Given the push for coherence in teacher understanding, it seems critical to more carefully understand what resources teachers have, which are invoked in different kinds of settings, how to support teachers in creating new connections within their existing understandings, and what understandings may need to be developed.

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References

Baumert, J., Kunter, M., Bum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., …, & Tsai, Y. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133-180.

Bédard, J., & Chi, M. T. H. (1992). Expertise. *Current Directions in Psychological Science, 1*(4), 135-139.

Begles, E. G. (1975). Critical variables in mathematics education: Findings from a survey of the empirical literature. Washington, DC: Mathematical Association of America & National Council of Teachers of Mathematics.

Ben-Chaim, D., Keret, Y., & Ilany, B.-S. (2012). *Ratio and proportion: Research and teaching in mathematics teachers' education*. Rotterdam, The Netherlands: Sense Publishers.

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and irresistibility of secondary students' errors. Educational Studies in Mathematics, 50(3), 311-334.

diSessa, A. A. (1988). Knowledge in pieces. In G. Forman & P. Pufall (Eds.), *Constructivism in the computer age* (pp. 49-70). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

diSessa, A. A. (2006). A history of conceptual change research: Threads and fault lines. In R. K. Sawyer (Ed.). The Cambridge handbook of the learning sciences (pp. 265-282). New York: Cambridge University Press.

Empson, S. B., Greenstein, S., Maldonado, L., & Roschelle, J. (2013). Scaling up innovative mathematics in the middle grades: Case studies of "good enough" enactments. In S. J. Hegedus & J. Roschelle (Eds.), *The SimCalc vision and contributions: Democratizing access to important mathematics*, (pp. 251-270). New York: Springer.

Ginsburg, H. P. (1997). Entering the child's mind: The clinical interview in psychological research and practice. Cambridge, UK: Cambridge University Press.

Goe, L., & Stickler, L. M. (2008). Teacher quality and student achievement: Making the most of recent research. Washington, D.C.: National Comprehensive Center for Teacher Quality.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. American Educational Research Journal, 42(2), 371-406.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction, 26*(1), 95-143.

Karplus, R., Pulos, S., & Stage, E. K. (1983). Early adolescents' proportional reasoning of "rate" problems. *Educational Studies in Mathematics, 14*, 219-233.

Lamon. S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (629-668). Charlotte, NC: Information Age Publishing.

Lobato, J., & Ellis, A. B. (2010). *Essential understandings: Ratios, proportions, and proportional reasoning*. In R. M. Zbiek (Series Ed.), *Essential understandings*. Reston, VA: National Council of Teachers of Mathematics.

Ma, L. (1999). Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States. Mahwah, NJ: Lawrence Erlbaum Associates.

National Mathematics Advisory Panel (NMAP) (2008). Foundations for success: The final report of the National Mathematics Advisory Panel. Washington, DC: U. S. Department of Education.

Orrill, C. H., & Brown, R. E. (2012). Making sense of double number lines in professional development: Exploring teachers' understandings of proportional relationships. *Journal of Mathematics Teacher Education, 15*(5), 381-403.

Post, T., Harel, G., Behr, M., & Lesh, R. (1988). Intermediate teachers knowledge of rational number concepts. In Fennema, et al. (Eds.), Papers from First Wisconsin Symposium for Research on Teaching and Learning Mathematics (pp. 194-219). Madison, WI: Wisconsin Center for Education Research.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. Journal of Mathematics Teacher Education, 11, 499-511. DOI: 0857-008-9089-5

Singh, P. (2000). Understanding the concepts of proportion and ratio constructed by two grade six students, *Educational Studies in Mathematics, 43*(3), 271-292.

Smith, J. P., diSessa, A. A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences, 3*(2), 115-163.