HOW PRESERVICE TEACHERS RESPOND TO STUDENT-INVENTED STRATEGIES ON WHOLE NUMBER MULTIPLICATION

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Although students' invented strategies typically prove to be effective in the improvement of students' mathematical understanding, little is known about how preservice teachers interpret and respond to student-invented strategies on whole number multiplication. This study investigated the nature of 25 preservice teachers' interpretations of and responses to students' correct and incorrect strategies for whole number multiplication. Results suggest that the mathematical depth of their responses and their consideration of student thinking differed based on the correctness of the student work. Implications for teacher educators and future researchers are discussed in accordance with the findings of the study.

Keywords: Teacher Knowledge, Teacher Education-Preservice, Number Concepts and Operations

Whole number multiplication is one of the most challenging operations in both calculation and justification in elementary mathematics education (Flowers, Kline, & Rubenstein, 2003). Researchers have documented that not only children experience difficulty in understanding the reasons for the standard algorithm, but also teachers. They also reported that students learn better when they are asked to create their own strategies different from the traditional method (Campbell, Rowan, & Suarez, 1998; Carroll & Porter, 1997; Huinker et al., 2003). NCTM (2000) suggests that teachers should spend a significant amount of time with student-invented methods that arise in a typical mathematics classroom and should think about how to help students build on them before introducing the standard algorithms, because students who invent strategies are involved intimately in the process of making sense of mathematics (e.g., Ball, 1988/1989). Ball (1989) also stresses that teachers need to address the aforementioned student-invented strategies as a window into student understanding and to endeavor to help students understand the conceptual thinking behind the mathematics. However, in a classroom situation where student-invented strategies are encouraged in learning mathematics, it is plausible that students make errors, and little is known about how to prepare preservice teachers (PSTs) for responding to these errors. The purpose of this study is to investigate PSTs' interpretations of and responses to students' correct and incorrect student-invented strategies involving whole number multiplication. Our purpose to uncover PSTs' ideas about teaching multi-digit multiplication and ideas about student-invented strategies is practically significant for teacher preparation programs. The research questions for the study were: (1) How do PSTs interpret correct and incorrect student-invented strategies with whole number multiplication?; (2) How do PSTs respond to correct and incorrect student-invented strategies with whole number multiplication?; (3) Do PSTs' interpretation and responses differ depending on the correctness of the student-invented strategies?

Theoretical Background

Research on Students' Strategies in Whole Number Multiplication

Numerous research studies on students' strategies involving whole number operations have been conducted not only in the US, (e.g., Carpenter et al., 1992; Carroll, 2000) but also in many other countries (e.g., Anghileri et al, 2002; Torbeyns et al., 2006). Table 1 shows some examples of student-invented strategies for multiplication reported from the literature with the example problem (28 x 7 = ?) (Bass, 2003; Carpenter et al., 1992; Carroll, 2000; Carroll & Porter, 1997; Huinker et al., 2003; Selter, 2002).

Table 1: Student-invented Strategies for Whole Number Multiplication

Strategies	**Characteristics**	**Examples (e.g., 28 x 7)**
1. Direct modeling	Use manipulatives or drawings to simulate the problem	
2. Repeated addition	Think multiplication as repeated addition	28 x 7 = 28 groups of 7 =7+7+7+....+7 =196
3. Chunking method	Chunk the addends or successive doubling	28 x 7 = 28 groups of 7 7 10s (70) + 7 10s (70) + 8 7s (56) = 70 + 70 + 56 = 196
3. Compensating (or varying method)	Round one of the factors to a multiple of 10 to make the multiplication easier and compensate by subtracting the extra.	28 x 7 → 30 x 10 = 300 → 300 – 14 = 196
4. Partial Products (or decomposition, or partitioning method)	Use the base-ten structure to break down the factors into partial products and use the distributive property	28 x 7 = (20 + 8) x 7 = (20 x 7) + (8 x 7) = 140 + 56 = 196

Among the student-invented strategies, partial product and compensation strategies were chosen in this study for three reasons—(1) they are frequently used by students, (2) they are mathematically efficient, and (3) they can be difficult to use with large numbers and are thus related to student error. First, these strategies are common strategies students invent on their own when they are asked to solve problems involving multiplication (Carpenter, Fennema, & Franke, 1992). Moreover, Bass (2003) asserts that these can be considered algorithms for whole number multiplication because they involve "a precisely specified sequence of steps" that can be programmed to always produce the correct solution (p. 324). However, these methods would be become cumbersome and difficult to use when applied to a problem involving larger numbers. In particular, without sound understanding of compensation, some students tend to apply these methods incorrectly (Schifter, Bastable, & Russell, 1999). For example, for the problem 28 x 7, a student called Tommy changed 28 x 7 into 30 x 10 and then took away 2 and 3 since he added 2 to the 28 and 3 to the 7. In the use of the compensation strategy, the ability to make an adjustment or compensation is important. In Tommy's case, he increased the 2 groups of 10 and 3 groups of 28. When students make errors, teachers should be able to provide an appropriate intervention. However, there is little research on how PSTs would respond to student-invented strategies, particularly, the compensation strategy on whole number multiplication. Therefore, this incorrect student-invented strategy was chosen for the study.

Research on Teacher Knowledge and Approaches

Researchers have studied the ways teachers understand content knowledge (CK) and/or pedagogical content knowledge (PCK) in several mathematics content areas including whole number operations (e.g., Ball & Bass, 2000; McClain, 2003; Thanheiser, 2009), fraction operations (Author, year 1), and other content areas and the relationship between teachers' CK, PCK, and their teaching practices (e.g., Ball, 1990, Krass et al., 2008; Prediger, 2010; Author, year 1; Author, year 2). These studies commonly reported that many US teachers hold narrow, procedural understandings of algorithms. In addition, they reported that PCK was highly correlated with CK mastery, thus suggesting that deep knowledge of the subject matter is a critical precondition for PCK (Baumert et al., 2010). Furthermore, they reported that teachers with more PCK display a broader repertoire of teaching strategies for creating cognitively stimulating learning situations (Ma, 1999). Much of this research, however, has focused on *teachers' knowledge* on CK, PCK, and teaching practices on common mathematical concepts focusing on *traditional methods* (or standard algorithms). As a result, little is known about the implications in teacher education programs for developing PSTs' responses to student-generated strategies. This article is intended to address this gap.

Methods

Twenty-five preservice teachers participated in the study from an elementary mathematics methods class at a large southeast university in the United States which utilizes the Holmes' model of teacher preparation. All participants had completed a required mathematics course equivalent to a 3 credit pre-algebra course either in their freshman or sophomore years. The mathematics methods class, which lasted approximately 14 weeks, was taught by the first author. The course was designed to support PSTs' understanding of approaches that are relevant to the teaching and learning of mathematics, particularly in the elementary grades. During each lesson, PSTs were involved in analyzing children's work through discussion of several samples in small groups and then as a whole class.

The main task used in this study presents student-invented strategies through a classroom scenario in which two hypothetical students come up with different solution methods to a two-digit whole number multiplication problem. The task consisted of three questions (see Figure 1) that took about 30 minutes to complete. The task was developed based on actual elementary students who appeared on a video from the *Developing Mathematical Ideas* curriculum (Schifter, Bastable, & Russell, 1999). The first student, Tommy, used a student-invented strategy, compensation, but failed to execute the procedure correctly. Dan, the second student, used a different student-invented strategy, decomposition (partial products), and produced the correct answer.

<table>
<tr><td colspan="2">You are teaching whole number multiplication problem 28 x7 to fourth graders. You asked students to solve it. After a few minutes, two students came to the board and explained their methods in a following way</td></tr>
<tr><td>Tommy's strategy
$\begin{array}{r}28\\ \times 7\\ \hline\end{array} \Rightarrow \begin{array}{r}30\\ \times 10\\ \hline 300\end{array} \Rightarrow \begin{array}{r}300\\ -\ 2\\ \hline 298\\ -\ 3\\ \hline 295\end{array}$</td><td>Dan's strategy
28×7
$7 \times 20 = 140$
$7 \times 8 = 56$ > 196</td></tr>
<tr><td colspan="2">1. Explain the logic behind each student's procedure and describe why you think each strategy will or will not work for all whole numbers.
2. How would you respond to Tommy? Describe your intervention using pictorial models / drawings/ numerical expressions. Explain it as much detail as you can.
3. How would you help Dan to develop the multiplication traditional algorithm from his method? Show the connection between drawings and numerical expressions to explain your teaching strategy.</td></tr>
</table>

Figure 1: Pedagogical Task: "How Would you Respond to Tommy and Dan?"

Martinez, M. & Castro Superfine, A (Eds.). (2013). *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Chicago, IL: University of Illinois at Chicago.

The task described above was administered as a survey to the entire class in two methods course sections towards the end of the fall semester of the 2009-2010 academic year. Table 2 shows an overview of the analytical framework associated with each task for the study. Data analysis involved five processes: (1) an initial reading of each PST's response, (2) identifying correctness of the responses, (3) exploring the subcategories under each analytical aspect according to the framework, (4) coding the categories and subcategories, and (5) interpreting the data quantitatively and qualitatively (Creswell, 1988). For example, responses to the first question, after having been identified based on correctness, were analyzed by looking at whether PSTs pointed to the underlying concepts or important properties related to each student's method. Using the conceptual versus procedural distinction guided by the work of Rittle-Johnson and Alibali (1999), a four-point rubric provided a framework for the different depth levels of the preservice teachers' justifications to each student's method.

Task	Item	Analysis aspects	Method
CK	Q1: Student method	• Validity/ Generalizability	Correctness
	Q1: Justification	• Conceptual vs. procedural	Develop a scoring rubric
PCK	Q2-3: Intervention	• Conceptual vs. procedural	Identify teaching category
		• Forms of address—Teacher vs. student-oriented	Develop categories
		• Type of cognitive process—Cognitive status vs. cognitive action	Frequency/Quantiative analysis
		• Type of model and contextual problems(Set, area, concrete, pictorial, etc.)	Frequency/Quantiative analysis

Figure 2: Methods Used for Analyses

The same procedure was used for the analysis of PSTs' responses to the second and third questions. Each code was then documented in Excel according to the task sub-domains, and a data table containing all the categorized responses for each participant was developed. In particular, in order to determine whether and how PSTs respond differently depending on the correctness of the student-generated strategies (research question 3), the Fisher exact value test, which is a non-parametric statistical significance test to determine dependent relationships in contingency tables, specifically for small sample sizes (Ott & Longnecker, 2001), was used for comparisons among 3 variables: (1) the definition of multiplication used in interpreting student logic, (2) depth of reasoning when interpreting student logic, and (3) the use of model-type when responding to students. In addition, we used a Z-test for independent proportions for the analysis of how PSTs' consideration of student thinking might differ based on students' correctness. Furthermore, for the analysis of PSTs' choices to respond with teacher-centered or student-centered responses, a chi-square test was used to determine if there is a dependent relationship between correctness of student work and student-centeredness of the PST's response.

Summary of Results

We found that most of the PSTs correctly recognized the validity and generalizability of Tommy's strategy and Dan's strategy. However, when it came to justifying the reasons behind the procedure and providing good intervention, around half related concept to procedure using meaningful understanding of multiplication. The PSTs mostly relied on the 'show and tell' approaches. In addition, this study provided evidence, with a significance level of 0.05, to support that the following three variables depended on whether the student work is correct or incorrect: (1) the definition of multiplication used by PSTs, (2) the depth of explanation PSTs gave, (3) how much PSTs talk about student thinking. In this paper, due to the space limit, we answer only the first and third research questions in detail.

Preservice Teachers' Interpretations of Student-invented Strategies Interpretations of Tommy's logic. All 25 participants correctly identified Tommy's strategy as not generalizable to all whole numbers. The PSTs' justifications were analyzed with respect to the following: (1) how to explain the process associated with Tommy's strategy, (2) what is the definition of multiplication used, and (3) what is the depth of justification. With respect to the definition of multiplication, we observed most made use of the equal groups definition of multiplication, while some responses focused on procedure more than on concepts and did not refer to either meaning of multiplication. A majority, 60%, of the PSTs used the equal groups definition of multiplication in their interpretations of Tommy's strategy, while only 2 participants, 8%, used the area definition of multiplication. 32% of the PSTs chose to address neither the area nor the equal groups definition, but chose to explain Tommy's logic referring mostly to the procedures Tommy used.

Table 2: Depth of Interpretations of Tommy's Logic

Rating	Description of response type	Example	Freq. (N = 25)
4	Clear and convincing explanation that includes discussion on the concept of multiplication. Errors are nonexistent.	Tommy tried to make the numbers easier to work with. He added 2 to 28 to make 30. Then added 3 to 7 to make 10. He subtracted the 2 and 3 he added, but when he added 2 to 28, he turned it into 30 groups of 7 and when he added 3 to 7 he added 3 more to each group. Tommy's error is not realizing the value of the numbers he is adding. To make this strategy work he could try Adding 2 means 2 more groups of 7. $28 \times 7 \Rightarrow 30 \times 7 \Rightarrow 210 - 14 = 196$	32% (8)
3	Understandable, but less-detailed, explanation that demonstrates some conceptual knowledge of multiplication. Errors are minimal.	This strategy will not work. Tom is thinking that since he added two more to 28, he can just take 2 away and 3 away because he added it to the 10. He does not understand that those 2 and 3 created little pieces contributed towards the area. I would use the area model to explain this problem.	24% (6)
2	Clear explanation, but one that does not give appropriate reasoning why the method is not generalizable.	Tommy has come up with a good strategy but in his rounding up to do this problem when going back to subtract the amount he rounded up he then applies the addition strategy. He obviously has not gained the understanding of groups in multiplication. His strategy works with addition and subtraction of whole numbers but will not work with multiplication of whole numbers.	16% (4)
1	Unclear, incorrect explanation or one that does not address the question.	This strategy will not work unless either the multiplicand or the multiplier is a whole number. What works to make the multiplicand or the multiplier a whole number, then subtract away the average.	28% (7)

To analyze depth of responses, we used a four-point rubric which distinguishes responses of differing depths, ranging from clear and convincing explanations that include discussion of the meaning of multiplication to those that are less-detailed, unclear, or incomplete (see Table 2 above). The ratings for these interpretations have a mean of 2.6 and a standard deviation of 1.2, indicating a moderate measure of spread in the distribution. 32% of the participants' interpretations were ranked at the 4 level, and all of these 4-point interpretations utilized the equal groups definition of multiplication. Also, 24% of the interpretations of Tommy's logic ranked at the 3 level. Among these 6 participants who scored 3 points, 3 used the equal groups definition, 1 used the area definition, and 2 chose to focus on procedure rather than either definition of multiplication. Ranking at the 2 level were 16% of the interpretations, which were clear but did not give appropriate reasoning why Tommy's method was not generalizable. Lastly, 28% participants gave unclear or incorrect interpretations, scoring a 1 on our four-point rubric.

Interpretations of Dan's logic. All participants correctly identified Dan's correct strategy as generalizable to all whole numbers. To analyze definitions of multiplication used to interpret Dan's thinking, we used the same categories observed in the analysis of their interpretations of Tommy's logic--equal groups definition, area definition, and a focus on procedure. We found a different tendency in the use of the definitions. With the interpretations of Tommy's logic, most of the PSTs addressed one of the definitions of multiplication; however, when interpreting Dan's logic, an overwhelming majority chose to only focus on procedure and failed to relate Dan's logic to any definition of multiplication. More specifically, 92% of the PSTs (23 out of 25) failed to connect Dan's logic with any definition of multiplication. Only 8% made a reference to the equal groups definition, such as "Dan's logic is that 28 groups of 7 is equal to 20 groups of 7 plus 8 groups of 7." Also, none mentioned the area definition. The 4-point rubric was used to analyze the depth of the interpretations of Dan's logic (see Table 3). Most of the interpretations were rated as either a 2 or a 3. This indicates that most responses could have been more detailed or could have better explained the generalizability of Dan's method.

Table 3: Depth of Interpretations of Dan's Logic

Rating	Description of Response Type	Example	Frequency (N = 25)
4	Clear and convincing explanation that includes discussion on the concept of multiplication, instead of solely on multiplication procedures. Errors are nonexistent.	Dan's strategy was to multiply the seven by the tens place of the other number and to multiply by the ones place. Then he added those 2 numbers together. This strategy works because he has not changed the problem, but simply isolated the two place values to simplify it.	4% (1)
3	Understandable, but less-detailed, explanation that demonstrates some conceptual knowledge of multiplication of whole numbers. Errors are minimal.	Dan breaks apart 28 into 20 and 8 but still multiplies 20 and 8 by 7, which will work for all whole number as long as all place values are represented.	40% (10)
2	Clear explanation, but one that does not give appropriate reasoning why the method is not generalizable.	Dan is breaking his numbers down so that they are easier to work with. Once he completes his simplified problem he adds those numbers together. His strategy will work for all whole numbers.	52% (13)
1	Unclear, incorrect explanation or one that does not address the	Dan broke 28 into twenty and eight because they are easier numbers to multiply. Then he	4% (1)

	question.	added them together. This strategy does not work for all whole numbers.	

Whether responses differ depending on the correctness of student strategies. In the interest of uncovering how PSTs make their decisions, our third research question asks if there were observable differences in PSTs' interpretations and responses based on the correctness of the student-invented strategies. We performed statistical tests to compare responses to Tommy's incorrect method to responses to Dan's correct method, in particular, with respect to the following five aspects—(1) differences in definition of multiplication used, (2) differences in depth of interpretation, (3) differences in discussion of student thinking, (3) differences in student-centeredness of responses, and (4) differences in type of model used.

First, results from the Fisher Exact Value test suggest a statistically significant difference in PSTs' use of the definition of multiplication with Tommy and Dan ($p = 0.00028$). In particular, we found that the equal groups definition of multiplication was used more often when interpreting incorrect logic. Interestingly, when interpreting correct logic, the PSTs tended to give purely procedural interpretations that were void of any definition of multiplication.

Second, results from the Fisher Exact Value test showed that the depth of explanation given by a PST is dependent on whether interpreting correct or incorrect student strategies ($p = 0.00093$). To discover the nature of this dependence, we compared the frequency on the 4-point rubric shown in Tables 2 and 3. When interpreting an incorrect student-invented strategy, there is more variance in the depth of responses, with more ratings at the highest or lowest end than between. On the other hand, when responding to a correct student-invented strategy, there is less variance, with most of the responses rating in the middle of the rubric and with very few responses at the highest or lowest end. This means that there were more high-quality, in-depth interpretations given of incorrect student strategies than correct student strategies.

Third, results from the z-test reveal that discussion of student thinking is dependent on students' correctness (a p-value of 0.0099, $Z = 2.579$). Although the task only asked how the PSTs would respond to Tommy and Dan, we noticed that many of them chose to use some language about students' thinking within their responses. After noticing that many of the PSTs chose to use this kind of cognitive language, we analyzed their choice of words and phrases. The categories that emerged from this analysis were similar to those of Sfard (1998). We sought vocabulary cues that frame learning either as a cognitive process or as *cognitive status*. For example, if there is think-action taking place, with words like, *recognize*, *see*, *forget*, or *remember*, PSTs' responses were categorized as describing math learning using a cognitive process. On the other hand, a response was categorized as describing math learning using a *cognitive status* if there is acquisitional language, such as "gain understanding," "gaps in knowledge," or "imparting ideas," portray math knowledge as a *cognitive status*, rather than *cognitive action*. We found that sixty eight percent (17 out of 25) were categorized as cognitive action while the rest of the participants showed views of cognitive status when talking about Tommy's learning. However, interestingly, the PSTs gave more neutral responses, not using many words that referred to Dan's thinking. Only two participants used cognitive action wording, which portrayed learning as engagement, and the same number of participants used wording that portrayed knowledge as a substance to be acquired. Results from the z-test suggest that a difference exists in the proportion of PSTs discussing student thinking based on whether the student's solution is correct or incorrect.

Forth, results from a chi-square test suggests that there is no statistical difference in PSTs' responses to Tommy and Dan with respect to student-centeredness ($p = 0.765$, $\chi^2(1) = 0.089$). This suggests that student-centeredness is independent of students' correctness.

Fifth and lastly, we also found that the types of models PSTs used are not dependent on the correctness of the students' work. The Fisher Exact Value test showed no such significant relationship ($p = 0.876$), suggesting that PSTs choose certain types of models to use in their responses, regardless of whether the student to whom they are responding is correct or incorrect in their reasoning.

Discussion and Implications

The findings of this study have implications regarding both CK and PCK for preservice teachers. For example, regarding CK, consistent with the findings from previous studies (e.g., Hill & Ball, 2004; Son & Crespo, 2009), this study stresses the importance of gaining a deeper understanding of the meaning of multiplication requires justifying of how and why an answer works and requires connecting models to computations in responding to students. Teacher education programs should pay more attention to strengthen PSTs' mathematical base and their abilities to provide justification in terms of not only traditional methods for whole number operations but also student-invented strategies. However, different from previous studies, this study investigated CK focusing on correct and incorrect student-invented strategies, and results provides further information about preservice teachers' tendency such as what type of scenario elicits such in-depth justifications and connections. For example, we found that incorrect student work samples elicit more in-depth justifications, whereas choice of model is not influenced by the correctness of the work. These findings suggest teacher educators might use incorrect student-invented strategies to promote the justifications and connections that promote CK.

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