# STUDENTS' CONCEPT IMAGES OF AVERAGE RATE OF CHANGE

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*In this paper, we describe students' concept images of average and average rate of change and the similarities and differences between those concept images. We do so by describing the students' ways of thinking and ways of understanding average and average rate of change, and how the students' meanings for average influenced their conceptions of average and instantaneous rates of change. We describe the importance of everyday meanings for average in students' conceptions of rate, and propose how instruction might be tailored to address this link. We conclude by discussing implications of this work for teaching average and instantaneous rates of change in single and multivariable calculus, and suggest important directions for future research.*

Keywords: Advanced Mathematical Thinking, Cognition, Cognition, Learning Trajectories (or Progressions)

## Introduction

The purpose of this paper is to explain how students leverage their understanding of average in their conception of average rate of change in differential and multivariable calculus and the implications of that leveraging for ideas that rely on a concept of average. Rate of change is foundational to calculus because it allows a student to represent how fast a quantity changes with respect to one or more other quantities. While average rate of change has a specific mathematical meaning in calculus, the word *average* may have lexical ambiguity because of its use in statistics and everyday language (Barwell, 2005). We hypothesized that students' understanding of average created confusion as they learned about average rate of change in calculus, and that they developed meanings for average rate of change that relied on an everyday understanding of the word average. We sought to characterize what students' concept images for average rate of change by focusing on their ways of thinking about average and their subsequent ways of thinking about average and instantaneous rate of change. We use *ways of thinking* as Harel and Koichu (2010) do to mean "a cognitive characteristic of a person's ways of understanding associated with a particular mental act" (Harel & Koichu, 2010, p. 117). In this paper, we describe the theoretical underpinnings of the study, identify how our assumptions about student thinking drove the study's design, illustrate our methodology and coding, and present a framework that characterizes students' concept images of average and average rate of change. We argue that specifically addressing the different uses of average in mathematics, statistics, and everyday language is crucial to students developing a coherent understanding of average and instantaneous rate of change in calculus and propose ideas for helping students develop the conceptions of average and instantaneous rate of change that we intend.

## Theoretical Framework

In this study, we focused on representing students' understanding and thinking, which by their nature are models that are shaped by our inferences based on students' actions and words. This study relied on the assumption that ways of thinking and ways of understanding (Harel &

Koichu, 2010) reside and develop at the level of the individual and reveal themselves in the decisions and actions students undertake.

Given our assumptions about learning and thinking, we used Vinner's (1983) definition of *concept images* as an orienting framework because students' understandings of average were largely imagistic in nature. We drew on Vinner's (1983) definition of *concept image* as the set of properties associated with a concept together with the all the mental pictures the student has ever associated with the concept. Vinner (1983) differentiates between the concept image and the concept definition (the verbal definition typically used to introduce a concept) and proposes that while handling a concept requires both image and definition, "in thinking, almost always the concept image will be evoked" (p.293). That is, while engaging in mathematical thinking, students tend to use their mental pictures of a concept rather than a symbolic or verbal definition.

We analyzed students' concept images for average and average rate of change. We looked for and asked specifically about similarities in how *average* was used in find-the-mean computational problems and in questions about two- and three-variable functions' average rates of change. We looked for students' *ways of understanding* both average and average rate of change and their *ways of thinking* about these two topics. We make the same distinction between *ways of understanding* and *ways of thinking* that Harel and Koichu (2010) do. That is, a *way of understanding* is the product of a mental act – a single moment as a student grapples with a mathematical situation, while a *way of thinking* "a cognitive characteristic of a person's ways of understanding associated with a particular mental act" (Harel & Koichu, 2010, p. 117). By 'characteristic,' Harel and Koichu mean a distinguishing trait. A *way of thinking* is a pattern of ways of understanding.

The result of analyzing students' concept images under the framework of *ways of understanding* and *ways of thinking* are descriptions of how students think about average and average rate of change, expressed imagistically as they are in students' minds.

## Literature Review

Understanding rate of change is foundational to ways of thinking about ideas in calculus, yet many students possess difficulties reasoning about rate (Carlson et al., 2001; Rasmussen, 2000; Thompson & Silverman, 2008). Students' difficulties understanding rate of change include problems interpreting the derivative on a graph (Asiala et al., 1997) and focusing on cosmetic features of a graph (Ellis, 2009). Thompson (1994) found that the difficulties students displayed in understanding the fundamental theorem arose from impoverished concepts of rate of change and incoherent images of functional covariation. Thompson described a coherent way of thinking about average rate of change of a quantity as, "if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, then we would end up with the same amount of change in the dependent quantity as actually occurred"PAGE # ?. However, we observed that this way of understanding was difficult for students to achieve.

We hypothesized this difficulty with average rate of change could be attributed to meanings students associate with the word *average*. Students' meanings for words used in technical domains are connected to past experiences with the word (Lemke, 1990). *Average* is used both in everyday language and in mathematics and thus may have lexical ambiguity, or multiple meanings (Barwell, 2005). Meanings statistics students hold for *average* include "definitions that were not indicative of the center and…responses that were not obviously connected to the idea of average as a measure of center or what is typical" (Kaplan et al., 2009, p.11). Given these meanings for average and our hypotheses about students' use of average, we investigated calculus students' meanings for average in the context of compute-the-mean tasks. We then had students complete average rate of change tasks to see if their meanings for average affected their understanding of average rate of change.

## Method

### Subjects and Setting

We recruited sixteen multivariable calculus students from a pool of volunteers from six sections of multivariable calculus. We chose this course because it was the students' first exposure to functions of more than one variable in mathematics. This allowed us to observe the students' initial fits and starts with systems with more than one quantity, and to adjust our subsequent questions to more clearly understand their thinking. Each student participated in a pre and post interview. The pre and post interviews questions were designed to gain insight into the students' ways of thinking about function and rate of change. The pre-interview questions were open-ended and focused on single-variable functions and rates. The post-interview questions were also open-ended and consisted of questions about both single and multivariable rates of change.

### Analytical Method

Data analysis was multi-phased. We used the pre-interviews to characterize ways of thinking about and understanding function and rate of change. We identified common behaviors and responses across interviews using grounded theory (Corbin & Strauss, 2008) with a particular focus on students' concept images for average rate of change and instantaneous rate of change. Our analyses from the pre-interviews suggested that students relied on colloquial definitions of average in their representations of average rate of change, and that those definitions were prevalent in both two and three dimensions. We designed the post-interviews to gain insight into students' images for average, average rate of change, and their thoughts about how those uses of average were related. We identified and transcribed important passages that gave insight into students' concept images for average in its use in statistics and calculus. We identified a set of concept images for both average and average rate of change using open and axial coding (Figures 1 and 2).

| Category | Criteria |
|---|---|
| Normal, typical, mediocre, common | Student uses the word 'normal,' 'mediocre,' 'typical,' or 'common' to describe 'average.' |
| Mean | Student uses the word 'mean' to describe 'average' or as a synonym for 'average' |
| Median, middle, center, balance point | Student uses the words 'median,' 'middle,' 'center,' or 'balance point' or talks about the average as being the middle or center of the data |
| Overall summary, representative value, value used to compare, estimate, expected value | Students talk about the average as a number that presents an overall summary of the data; a number that is representative of all the data; the average as an estimate/approximation or expected value for a new data point; or talk about using the average to compare data |
| Mode, most common number | Student uses the word 'mode' as synonymous for 'average' or talks about average as the most |

| | common number |
|---|---|

**Figure 1: Students' Concept Images for Average**

| Category | Criteria |
|---|---|
| Arithmetic mean of slopes | Student talks about summing slopes and dividing by number of slopes summed whether it is a finite or infinite number of slopes is irrelevant to the student. |
| Expected or most common slope | Student uses the word 'most common', 'expected', 'typical' to describe average rate of change. The student expects the average rate of change provides information about 'all' of the slopes. |
| Constant rate of change | Student describes average rate of change as the constant rate of change required to produce the same change in the function over the original interval of input. |
| Smoothing out of all the slopes | Student describes the average rate of change as the all of the slopes smoothed out. Student describes decreasing the 'choppiness' of the slopes. |

**Figure 2: Students' Concept Images for Average Rate of Change**

## Results

The following results are representative of our findings for the sixteen students. We highlight three students' responses that represent the major categories of thinking and understanding we identified. We found that students carry their meanings for average into their thinking about average rate of change. We will demonstrate this with excerpts from student responses to the tasks shown in Figure 3. The interviews included two additional compute-the-mean tasks similar to question 1 and two additional conceptual rate of change problems similar to questions 2 and 3.

| **Questions** |
|---|
| (1) The data given below represent the masses of six fishing lures. What would the average mass of the lures mean? |
| (2) Suppose we define a function *f*, so that $f(x) = e^{-\cos(2^x)}$. Discuss the *process* you would use to determine the average rate of change of the function with respect to *x* over the interval [2.0, 2.2]. (3) Suppose we define a function f, so that $f(x,y) = e^{-\cos(xy)}$. Discuss the *process* you would use to determine the average rate of change of the function. What information do you need to know to complete this process? |

*Figure 3:* Representative Interview Tasks

Responses across these tasks demonstrate the use of concept images for average in the concept images for average rate of change. For instance, Brian's concept image for average is one of 'smoothing out,' a phrase he used explicitly when talking about average rates of change (Response to 2). We also infer that Brian's use of average rate of change being a constant rate as related to the idea of smoothing out; that is, the constant rate is a smoothed-out number made from finding the mean of instantaneous rates (Response to 3). Jordan's concept image for average included the property that an average is a 'typical value' (Response to 1) and she correspondingly thought about average rate of change as a typical slope (Response to 2). Jane's concept image for average included the property of an average being 'common' (Response to 1) and she thought average rate of change as the most common value (Responses to 2 and 3).

*Brian:* [Response to 1] I see the average as kind of like adding everything up into a big ball, and then smoothing it out into equivalent pieces.

[Response to 2] I see the average rate of change like a constant rate of change. Like, how fast the function would need to change to produce the same change in y over the same change in x, but at a constant rate. You take the change in y over the change in x, that kind of smooths it out to determine it for you.

[Response to 3] Now, well, this is harder but I still know I am finding a constant rate of change. However, to pick a constant rate, you have to specify a direction in space, or there would be infinite average rates of change. So, you still have a change in the function on top, but divided by a change one variable or the other. It tells you a constant rate of change.

*Jordan:* [Response to 1] Well, I sum the masses, then divide by how many there are, which tells me what their mass was mostly, or typically.

[Response to 2] Well, I am finding the slope between two points here Right, so I find the change in y over the change in x. That just tells me a typical slope.

[Response to 3] Again, I probably am finding a slope, an average slope, so I need a change in something over a change in something else. Probably a combination of z, x and y? Again, it would just tell me a typical slope.

*Jane:*[Response to 1] Well, it would be the most common mass of all of them, kind of giving me information about the seventh lure.

[Response to 2] The average rate of change tells me the most commonly occurring rate of change of all the rates of change, infinite of them in the interval. I find the change in y over change in x, and it tells me that, the most common value.

[Response to 3] Sure, I'm still finding an average rate, so I need something to divide into something else, probably change in z over a change in a combo of x and y. Gets me to the same point, the most commonly occurring rate of change. Like summing up all of the rates of change, and dividing by how many there are in the interval.

## Discussion

### Students' Concept Images

A concept image includes the properties of the concept, any mental pictures a student has ever associated with the concept, and a concept definition. Vinner (1983) writes that a concept definition is “a description of our concept image [and is] either taught to us or made up by us when we are asked to explain the concepts to somebody” (p.294). By ‘made up,’ Vinner means that students construct definitions as a result of their experiences with the concept.

We asked students to define ‘average’ and ‘average rate of change’ and the compare the meaning of the two, in addition to the questions in Figure 3. Many students expressed difficulty defining average (“everyone knows what average means”), and as a result we think their attempted definitions make good snapshots of their concept images. The majority of students’ concept images contained properties from everyday language and mathematics. For example, Jordan talked about an average mass as a typical mass and an average rate of change as a typical rate of change.

Mathematically, average may refer to mean, median, or mode (Triola, 2006) and students’ explanations reflected this ambiguity. Jane’s use of ‘most common’ and Jordan’s use of ‘what it is mostly’ are reflective of mode-as-average. Other students (excerpts not included) talked about average as a middle value or a balance point, reflecting average-as-median. While students described median and mode in their concept images for average, all students’ calculations for average mass were arithmetic means.

Students’ concept images for average rate of change reflected their concept images for average as a most common, typical, middle, or smoothed out value. For instance, Jordan talked about average as a most common value and talked about average rate of change as “the most commonly occurring rate of change of all the rates of change, infinite of them in the interval.” Brian talked about average as a smoothed-out value and described average rate of change as “You take the change in y over the change in x, that kind of smooths it out to determine it for you. I find the change in y over change in x, and it tells me that, the most common value.” While students frequently referred to computing average rate of change as taking the change in one variable divided by the change in another (e.g., Brian, Jordan, and Jane’s responses to 3), this seemed to be a rote procedure. That is, students did not seem to have an image of x,y and z as quantities or an image of a quantity changing with respect to another quantity at a constant rate.

We also identified a disconnect between the way students thought about average rate of change and their procedures: that is, they tended to think about it as a ‘smoothing out,’ a median, or a mode, but the procedure of change in one variable over change in another is more reflective of average-as-mean. We concluded that students were unsure when the different meanings for average were appropriate for a situation.

**Ways of Thinking and Understanding**

Harel and Koichu (2010) differentiate *way of understanding* as an in-the-moment process a student uses to construct meaning and *way of thinking* as a characteristic way of understanding. We looked at students’ ways of understanding average and average rate of change in computational problems, then looked for similarities in ways of understanding that would constitute ways of thinking. We found that students’ ways of thinking about average rate of change mirror their ways of thinking about average.

Students’ ways of understanding average included properties like normal, typical, mediocre, common, arithmetic mean, median, middle, center, balance point, mode, most common number, smoothed-out value, representative value and expected value (Figure 1). Their ways of understanding average rate of change included arithmetic mean, expected value, most common rate of change, typical rate of change, rate of change representative of all rates of change, and a smoothed-out rate of change (Figure 2). The overlap in ways of understanding across context indicate that students bring their way of understanding average to their way of understanding average rate of change. Moreover, these ways of understanding constitute ways of thinking for many students. We base this conclusion on the observation of patterns in students' ways of understanding across three compute-the-mean problems and four conceptual average rate of change problems.

We conclude that students' ways of thinking included a number of elements from ways of understanding an idea in different contexts. In the case of average rate of change, these ways of thinking seem to prevent students from thinking about average rate of change as a quantity changing at a constant rate with respect to another quantity.

## Implications

### Instantaneous Rate and Extensions

Our characterization of students' concept images for average and average rate of change has implications for how students think about instantaneous rate of change and for how they extend their way of thinking about rates of change to three dimensions. Many of the students who thought about average as an arithmetic mean determined that average rate of change was an arithmetic mean of a finite number of instantaneous rates of change. They could hold this conception in mind because they thought about an instantaneous rate as slope (a picture) without a measuring process attached to it. Thus, for these students an instantaneous rate was similar to the weight of a lure, and the average rate of change was an arithmetic average. This way of thinking ignores the limiting process in measuring a rate of change, and does not focus on measurement of quantities to determine how fast one is changing with respect to another. When the students attempted to determine how to interpret and measure rate of change in space, they relied on their image of instantaneous rate as a degree of slant of a line (without problematizing direction), and the average rate of change as an arithmetic mean of the degrees of slant. This way of thinking allowed the students to ignore the issue of direction in space and the limiting process that makes possible the computation of a derivative. These ways of thinking support a non-quantitative conception of rate of change for two reasons. First, the measurement of quantities is unimportant. Second, and partially because of the first, the limiting process is not necessary for the student. These issues confound the understandings we intend that students have, and likely constrain students from seeing rate as a quantification of how fast quantities are changing.

### Recommendations for Instruction

We propose that students must understand that average rate of change is a comparison of *change* in *quantities,* and that statistical average is a quantity that characterizes a number of quantities. We propose that the distinction between the uses of average can be supported using quantitative and covariational reasoning. (Thompson, 1994, 2011) In both statistic and calculus, students' understanding of average relies on quantities and their measures and students often associate the two because each uses division. However, average rate of change requires that students measure changes in quantities (quantitative reasoning), and compare those quantities to determine how fast one is changing with respect to another (a constant rate, requiring covariational reasoning). We believe that students must understand that while average rate of change and average in statistics use similar calculations, the result of those calculations represents different quantities. While space permits us from detailing a quantitative and covariational approach to teaching rate of change here, we have ideas for particular learning trajectory-type tasks that teach rate of change from these two perspectives.

## Future Directions

Our results suggest that students use their concept images for average to make sense of average rate of change. These concept images often lack the element that rate of change is a measurement of the change in one quantity with respect to the change in another quantity. This leaves students to impart different meanings to [f(b)-f(a)]/(b-a) and these meanings are based on their meanings for the word average. In other words, they apply concept images that are not appropriate for the situation. Moreover, not thinking about a rate of change as quantities changing results in students not seeing a need to make the change in one of those quantities approach zero: that is, there is no need in students' minds for a limiting process. We believe that students' development of the meanings instructors intend requires not just taking into account, but rather using productively, their concept images for topics related to new material. Having documented these concept images, the next step is to determine how instructors can use these to their productively in teaching calculus.

We hypothesize that a focus on quantitative and covariational reasoning may help students develop the intended meanings for average rate of change, the limiting process, and instantaneous rate of change. While the importance of quantitative and covariational reasoning has been highlighted in algebra and differential calculus, limited work has studied the role of these ways of thinking in multivariable calculus. Our work suggests that these ways of thinking are critical for upper-level calculus, and future work is needed to determine how to best foster these ways of thinking.

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