CAPTURING MIDDLE SCHOOL STUDENTS' UNDERSTANDING OF THE CONCEPT OF AREA USING VYGOTSKY'S CONCEPT FORMATION THEORY

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Incorporating Vygotsky's concept formation theory and Berger's appropriation theory, an assessment which highlights authentic operations with the concept of area was designed and administered to 44 sixth grade students to determine what types of understanding they maintained. The results indicated the more novel the situation, the more diverse stages of understanding were exhibited by children. Inconsistent levels of reasoning across different items were revealed for most individuals. We suggest that novel assessments grounded in concept formation theories may provide greater insights on children's understanding of mathematics.

Key words: Assessment and Evaluation, Measurement, Problem Solving

Introduction

Learners' prior domain knowledge of mathematics has long been identified as a key prerequisite for development of more sophisticated mathematical thinking (Shulman & Keislar, 1966; Bauersfeld, 1995; Lesh & Doerr, 2003). This connection has most prominently been voiced within the genre of research on mathematical problem solving (Schoenfeld, 1992), and understanding the interactions between domain knowledge and problem solvers' activities including strategy use, control, and beliefs has been the source of inquiry for over three decades (Lester, 1994). The body of existing literature points at a link between what learners know and the constraints that the existing knowledge imposes on their mathematical practices. While this point certainly merits attention, we are less convinced by how children's mathematical knowledge may have been characterized based on instruments used for capturing their understanding. Conventional assessment tools often focus on whether learners can use what they had presumably learned when confronted with tasks similar to those practiced. Such practice has led to production of numerous reports indicating learners' failure at transfer of knowledge when experiencing a new situation (Niss, Blum, & Galbraith, 2007). We posit that capturing, with some degree of accuracy, what children know demands research-based assessment instruments that reveal their ways of knowing. Of particular concern is not the use of instruments that capture what conventional knowledge children may have retained, but rather the particular types of understanding they hold. Currently, such instruments are rare in mathematics education (Adams, 2012).

Objectives of the Study

The purpose of the study reported here was twofold. First, we aimed to investigate the types of understanding of the concept of area that middle school seemingly held when tackling different problems. Second, we were motivated to determine the affordances of a theory driven instrument to make visible the various types of understanding children might have of the same topic. The following research questions were used to guide the study:

1. What stages of concept development are revealed through students' interactions with the tasks in terms of Vygotsky's concept formation theory?
2. What patterns of understanding of the concept of area do middle school students exhibit?

Theoretical Framework

In this study we incorporated two theoretical perspectives as groundings for our instrument development: Vygotsky's (1962) concept formation theory and Berger's (2004) appropriation theory. Vygotsky's theory proposes a framework for an individual's concept (word or sign) development within a social environment, while Berger's theory proposes an interpretation of Vygotsky's theory in the domain of mathematics by adding and omitting certain stages. Since Berger's theory was designed based on undergraduate students' performance in calculus and linear algebra, we conjectured that the developmental framework might be different for secondary students in other content areas. In order to benefit from her work without being conceptually restricted, both theories were utilized to inform this study.

According to these two complementary theories, concept development consists of three phases: heap, complex, and concept. In the *heap* phase, the learner associates a sign with another because of physical context or circumstance instead of any inherent or mathematical property of the signs. In the *complex* phase, objects are united in an individual's mind not only by his impressions, but also by existing bonds between them. However, the bonds between objects are concrete and factual instead of abstract and logical. The complex phase further contains the sub-stages described in the following paragraph.

During the *association complex*, the learner uses one mathematical sign as a nucleus and associates other signs with some common attributes based on objective and factual justification. For *chain complex*, the learner associates one mathematical sign with another based on some similarity and then links the new sign to another by a different attribute to form a chain. With *representation complex*, the learner identifies the visual or numerical representation of a mathematical object as the object itself. Properties abstracted from such representations are considered as the properties of the object. Students with a *pseudo-concept* could use and communicate the mathematical notion as if they fully understand it, although their understanding may be based on factual connections instead of logic.

Concept is defined as a mathematical idea with consistent and logical internal links (links between different properties and attributes of the concept) and external links (links of the concept to other concepts).

The tentative formation stages for the concept of area, as we conceptualized them, are illustrated in Figure 1. Examples of students' reasoning with area in Battista's (in press) Cognition-Based Assessment (CBA) were used as the main resource in referencing students' developmental understanding of the concept. Note that in the figure, underlined terms are the stages derived from Vygotsky's and Berger's theories, while Non-Measurement and Measurement (including Unit area and Formula) are the key components in the development of measurement reasoning identified by Battista's CBA levels. Additionally, each developmental stage of each component is followed by the corresponding examples from CBA levels as well as examples from our own experiences with students' reasoning (in *Italic*).

This framework guided the design of the assessment tool as well as the analysis of participants' responses in this study.

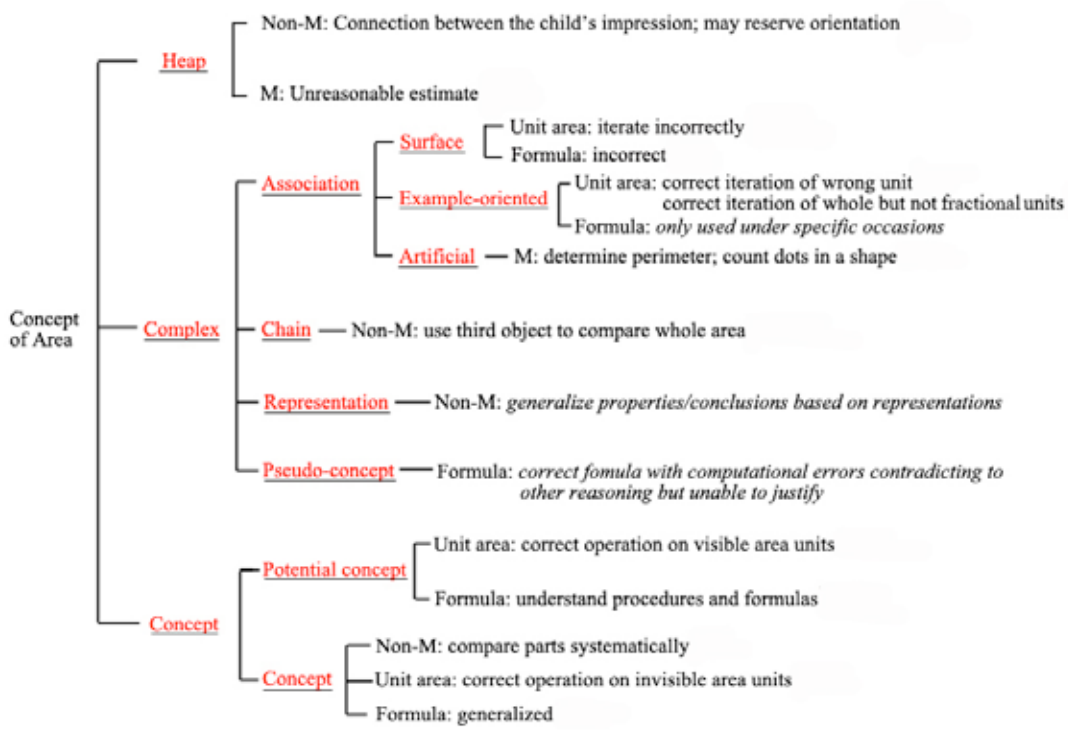

Figure 1. Tentative Developmental Stages of the Concept of Area

Methods

Participants

Participants included 44 sixth grade students from a mid-western suburban middle school. The students were from three distinct class periods of an algebra course taught by the same teacher at the time of data collection. They were observed by the lead author for 6 months prior to data collection. In the course of observations it became evident that they exhibited a range of different types and levels of understanding of the concept of area. These observations constituted the need to examine their thinking more carefully.

Contexts

The participants were given the assessment during one class period (50 minutes). Prior to administering the assessment, students were informed that they could use calculators if they felt they were needed. They were also reassured that if they felt they needed assistance when reading the problems, the researchers would provide assistance accordingly. Lastly, they were asked not to erase their work even if they considered it wrong.

Instrumentation

Five questions were selected from existing assessments and modified to resemble novel (non-textbook-like) tasks. Resources included items from TIMSS, CBA tasks, and Problem Sets from the Math Coaching Program at the Ohio State University. Table 1 shows the difference between a conventional task and its corresponding modified novel version produced for use in the study. The third column outlines the developmental stages expected to be elicited by the novel version.

Table 1: Difference Between Conventional Task and Novel Task

Conventional task	Novel task	Developmental stage and explanatory approach elicited by the novel item

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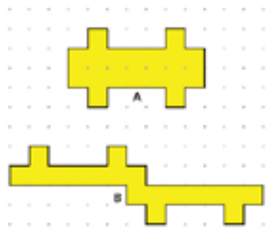
Which of these shapes has more area or room inside it, or do they have the same amount?  [CBA measurement task]	Which of these shapes has more area, or do they have the same amount? Explain how you reach your conclusion.  If you believe there's not enough information to answer the question, please state what information you would need and how you would use that information to answer it. [Item 1]	*Non-measurement reasoning:* Heap (compare based on impression) Chain complex (use third object to compare whole area) Concept (compare parts systematically)

The assessment items were aligned to associate with stages identified by the framework (with overlaps) where novel situations were created to provoke authentic interactions with the concept. Item 2 through 4 are displayed in the results section.

Analysis

Analysis of data followed three stages. First, two researchers independently reviewed all 44 sets of participants' responses to identify and document enacted approaches and coded developmental stages associated with each approach. Notes were compared for consistency in scoring. Children's approaches that were ambiguous or non-anticipated were discussed in the second step. The theoretical framework was adjusted based on the analysis of these responses; five more stages were identified and added to the original framework. Lastly, the distribution of developmental stages for each item was examined and potential patterns were abstracted.

Results

Table 2 summarizes the descriptions of different developmental stages associated with each item, and the number of times each stage appeared for each item. Stages in *italic* are those identified and added to the original framework based upon the researchers' initial analysis of students' responses. For Item 3, six students provided two types of reasoning in their responses, revealing two stages for each individual. Item 5 assessed general problem solving performance, which is not included in the table at this time. Blank answers are not included in the results.

Table 2: Number of Stages Elicited by Each Item

Stage	Description	Item 1	Item 2	Item 3	Item 4	Total
1.1	Heap - NonM: connection between the child's impression	9	3	0	0	13
1.2	Heap - M: unreasonable estimate	0	0	3	0	3
2.1.1.1	*Surface Association Complex – NonM: compare parts randomly*	2	0	0	0	2
2.1.1.2	Surface Association Complex – Unit area: iterate incorrectly	2	1	10	11*	24
2.1.1.3	Surface Association Complex – Formula: incorrect	0	1	2	0	3
2.1.2.1	Example-oriented Association Complex – Unit area: correct iteration of wrong unit or of whole but not fractional units	0	3	1	11	15
2.1.2.2	Example-oriented Association Complex – Formula: only use formula under specific occasions	0	0	0	0	0

2.1.3	Artificial Association Complex – M: determine perimeter; count dots in a shape	8	18	0	3	29
2.2.1	Chain Complex - NonM: use a third object compare whole area	1	0	0	0	1
2.2.2	*Chain Complex – M: use a third object to compare area*	0	3	0	0	3
2.3	Representation – generalize properties based on representations	0	0	0	0	0
2.4.1	*Pseudo-concept Complex – NonM: compare parts non-rigorously*	7	0	0	0	7
2.4.2	*Pseudo-concept Complex – M: empirical estimate*	6	0	1	0	6
2.4.3	*Pseudo-concept Complex – Unit area: Non-rigorous estimations on visible area units*	0	11	0	0	11
2.4.4	Pseudo-concept Complex – Formula: correct formula contradicting to other reasoning	0	1	0	0	1
3.1.1	Potential Concept – Unit area: correct operation on visible area units	0	1	0	7	8
3.1.2	Potential Concept – Formula: understand procedures and formulas	0	0	1	0	1
3.2.1	Concept – NonM: compare parts systematically	3	0	0	0	3
3.2.2	Concept – Unit area: correct operation on invisible area units	0	0	20	7*	27
3.2.3	Concept – Formula: generalized	0	0	0	0	0
Total		38	42	38	39	157

*Responses were from the copies where the grid was barely visible in item 4.

Three stages failed to be revealed in item 1-4: Example-oriented Association Complex – Formula (only use formula under specific occasions), Representation (generalize properties based on representations), and Concept – Formula (generalized). The Example-oriented Association Complex – Formula stage could be investigated by looking at both Item 3 and 5, while the latter two stages might be more likely to be elicited in a problem that allows extending and generalizing.

Figure 2 illustrates the distribution of the elicited stages for each item.

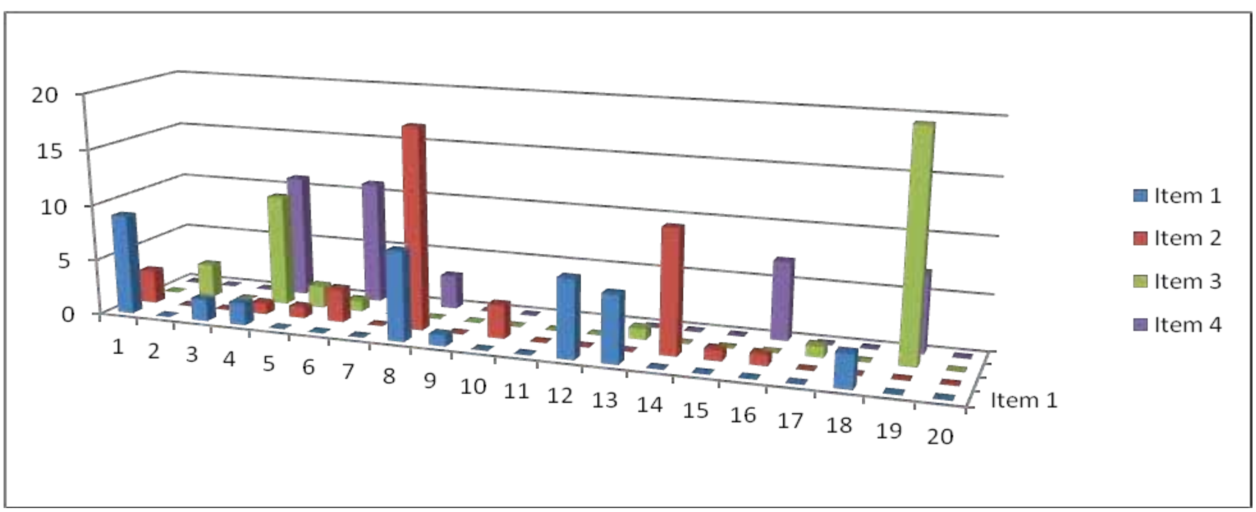


Figure 2: Distribution of Stages for Each Item

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As the graph illustrates, Item 1 and Item 2 provoked more diverse stages of knowing, while Item 4 elicited the most narrowed. Item 3, arguably the most conventional mathematical task among all items, dominantly stimulated the highest developmental stage for unit area; however, in our previous study (Zhang et al., 2010), the dominant approach (37%) for this problem among 292 fifth graders was to iterate incorrect triangles (stage 2.1.1.2).

A Closer Look at Learners' Approaches and Coding Implications

To expand on the previous quantitative account of aggregate responses, in this section we use one participant's written artifacts to each of the four items to demonstrate the range of analytical schemes children utilized when examining non-conventional tasks.

Student J's response to Item 1 is illustrated in Figure 3. Note that Item 1 asks the student to compare two irregular shapes without any measurement.

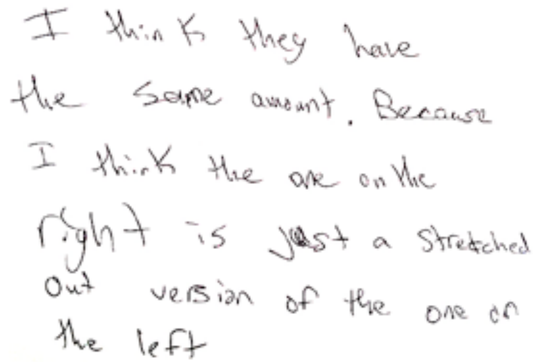

Figure 3: J's Response to Item 1

J's response to Item 1 was categorized as stage 1.1 (Heap – NonM) since the phrase "stretched out" did not provide enough concrete or logical mathematical evidence for his conclusion, despite the fact that his answer was correct. Since he did not provide any visual representation, we did not have sufficient evidence to rank his thinking beyond stage 1.1.

Item 2 along with J's response is illustrated in Figure 4.

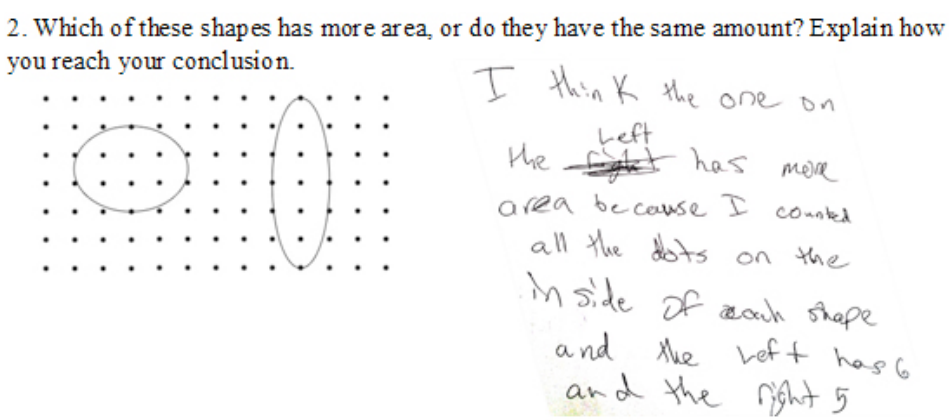

Figure 4: J's Response to Item 2

J's response to Item 2 was categorized as stage 2.1.3 (Artificial Association Complex – M) since he associated the area with the number of dots enclosed within the region. Some students considered not only the number of dots inside of each region, but also the ones "touching" the perimeter; those responses were also categorized into this stage, although the idea might be quite different from J's.

Item 3 along with J's response is illstrated in Figure 5.

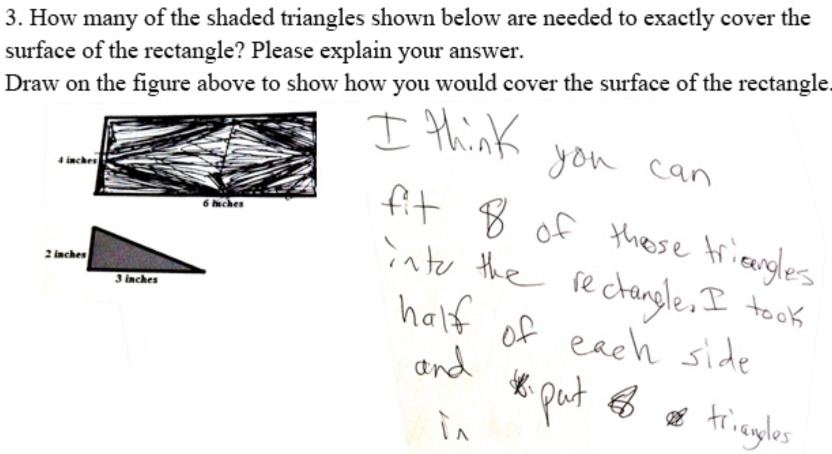

Figure 5: J's Response to Item 3

J's response to Item 3 was categorized as 3.2.2 (Concept – Unit area) since he correctly iterated eight shaded triangles in the rectangle. However, a part of his iteration was not very clear (the right bottom section of the rectangle). He may have iterated incorrectly during the process (stage 2.1.1.2), but the description he provided was valid.

Item 4 along with J's response is shown in Figure 6.

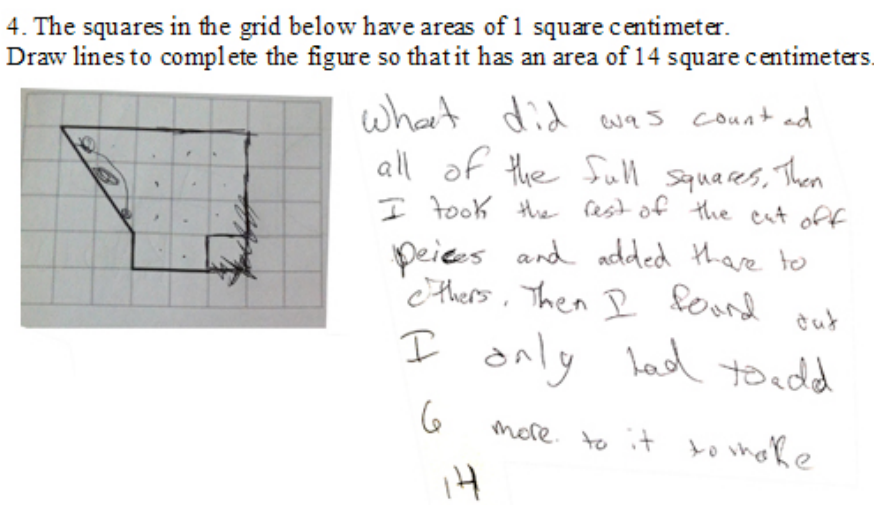

Figure 6: J's Response to Item 4

J's response to Item 4 was categorized as stage 3.1.1 (Potential Concept – Unit area) since he correctly paired up the four partial areas into two whole squares. Many students only paired up two partial squares while ignoring the smallest one and chose to draw a new partial area to make up the missing part (which was the part they ignored).

J's responses to Item 3 and Item 4 placed him at concept level of reasoning, but his responses to Item 1 and Item 2 were identified as heap or lower level complex reasoning. Such inconsistency was commonly observed among the participants. 25 out of 44 students showed

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concept level reasoning in Items 3 and/or Item 4 but *heap* or low *complex* level reasoning in item 1 and/or Item 2. 16 students exhibited consistent level of reasoning across all 4 items; 1 student showed *concept* level reasoning in Item 1 but low *complex* level reasoning for Item 2 to 4; 2 students showed inconsistent levels of reasoning which were different from the previous three patterns. A possible reason for such inconsistency is that individual's understanding for each component (Non-measurement, Unit area, and Formula) develops at different pace, and a higher level understanding of one component might be influenced/restricted by a lower level understanding of another component under novel situations.

Discussion and Conclusion

The major goals of the study were to examine the utility of a research based instrument grounded in theories of concept formation for revealing middle school children's conceptualization levels of area concept. Findings revealed that four of the items (1 to 4) successfully elicited 17 among the 20 developmental stages pertaining to the concept under study.

Findings suggest that stages and levels of understanding of a concept become far more visible when situations used for assessment are less familiar to what students may have experienced in textbooks. Relatively conventional situations appeared to elicit standardized approaches for solving problems, making the issue of assessing learning far simpler by categorizing them as right or wrong. Our findings suggest that novel assessments designed around concept formation theories may provide researchers greater capacity to articulate intricacies of children's understanding of mathematical concepts.

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