LINEAR ALGEBRA IN NEW ENVIRONMENTS (LINE)

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We report on some results from a multiyear development of new techniques and materials for teaching linear algebra. Our goals were to (a) to create a professional learning community across STEM disciplines, (b) to combine expertise in content and pedagogy in designing effective instructional practice, and (c) to use learning theories to support the conceptual alignment of content and pedagogical goals. In particular, our approach combines the use of domain-specific problems, APOS learning theory, and the development of the professional learning community. This set of practices was developed and deployed across four diverse institutions, in diverse linear algebra courses, and was effective across this diversity. We discuss the development of teaching materials, and present reflections from students and faculty which were shared during and after these "modules" were used in classes.

Keywords: Advanced Mathematical Thinking, Curriculum, Learning Theory, Post-Secondary Education, Problem Solving

Introduction

Linear algebra undergirds a remarkably diverse array of applications, such as computer graphics, network theory, economic and growth modeling, demographics, and cryptology. Indeed, it is this wide array of applications that makes linear algebra courses so popular among both mathematics and non-mathematics majors. Popular courses do not automatically enjoy effective pedagogy, however; improving the teaching and learning of linear algebra is the subject of ongoing research. Based on some of our prior investigations (Martin, Loch, Cooley, Dexter, and Vidakovic, 2010; Vidakovic, Cooley, Martin, and Meagher, 2008), we hypothesized that linear algebra pedagogy can be most effectively advanced through a strategy based on: (a) creating a professional learning community of STEM faculty dedicated to creating classroom experiences drawing from a diversity of application domains; (b) providing collaborative support, drawing on expertise in both content and pedagogy, for designing effective instructional practice; which, together (c) ensure conceptual alignment between content and pedagogical goals through the use of theories of learning and instructional models.

To test our hypothesis, we formed a community of investigation we call **LI**near algebra in **N**ew **E**nvironments (LINE). While LINE rests primarily on the three planks above, we also envision and support local change toward a reflective, collaborative culture of teaching and learning among STEM discipline faculty. Through our approach to linear algebra pedagogy, we model both (a) the integration of theoretical mathematical content, applications, and learning theories, and (b) co-teaching and collaboration among faculty with expertise in a variety of areas including mathematics, computer science (an applications domain) and mathematics education. We posit that this approach provides an enhanced experience for students, allowing them not only to achieve a deeper understanding of linear algebra but also to develop their ability to reflect on their own learning.

Background and Theoretical Framework

To connect application-specific content, pedagogical practice, and reflection, the LINE approach relies on an instructional methodology based on the Action-Process-Object-Schema (APOS) framework (Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996). APOS is a well-developed framework for understanding how students develop mathematical concepts; in addition to a model of cognition, it provides an instructional model intended to facilitate student development of richer and more sophisticated understandings of mathematical concepts. The approach has three components: It rests on a *theoretical analysis* (called a genetic decomposition) of what it means to understand a concept and how learners construct new concepts. This leads to the design of *instructional treatment* focused on these mental constructions. Then, as they teach, instructors *gather data*, which is analyzed in the context of the theoretical perspective and used to revise both theory and instruction, as needed.

The faculty in the LINE project, working as a professional learning community (Hamos, Bergin, Maki, Perez, Prival, Rainey, Rowell & VanderPutten, 2009), began the project with a reading seminar in which the faculty jointly examined mathematics learning theories—including APOS and other theories of scientific learning as well as cognitive difficulties related directly to linear algebra (Sierpinska, Drefus & Hillel, 1999) and collaborated in the initial development of modules for linear algebra using approaches based on APOS. The professional learning community continues to foster faculty development through a combination of activities among the different colleges using videoconferencing and working meetings at each other's institutions. Faculty also co-teach the modules, videotape each other while teaching, observe student engagement, and use these videotapes and observations to reflect on the learning and teaching processes in the context of learning theories.

Sample Module

Due to page restrictions, we discuss two representative "modules" to illustrate LINE principles rather than present them. Sample modules can be accessed from the authors. We chose the first one as an application in linear algebra and the second one as a more conceptual, theoretical module. After each module, we share some observations of the students' learning experiences.

There are four colleges participating in line: an urban public university in a major Northeast city with about 15,000 students, a land grant public university in the Northwest with about 30,000 students, a private college in the Midwest with about 2000 students, and an urban public university in a major Southeast city with about 30,000 students. Across these varying settings, though, the modules developed share important commonalities. In accordance with the line perspective, at least two faculty members developed and implemented the modules at each site. The faculty members work collaboratively at each site, and across sites, to develop the modules and integrate them into classes. Most importantly, the modules share a common aim: to introduce topics in ways that bring students to reflect on what they are learning and to actively readjust their understanding as they are exposed to new concepts, building on what they know and using that knowledge to grasp new ideas.

Sample Module I: Dot Product Application

This module was given to students after they had been introduced to the concept of the dot product and its connection to the cosine of the angle between two vectors. The module uses the dot product to assess if certain texts were written by a particular author by having students create frequency vectors based on "function" words, such as but, an, the, etc. and to find the similarity between these frequency vectors and those frequency vectors from a known original text by the same author in question.

Observations of Student Learning

The whole class discussed the module after it had been completed. Most students were able to successfully decouple the idea of similarity from magnitude. There was some student debate over the question of which vector would be "least similar" to u, with some students holding that –u was least similar (since it was in the opposite direction), and others holding that the perpendicular was least similar. One of the more important misconceptions that emerged from this discussion was that several students talked about using the dot product to measure the angle of *a* vector, rather than the angle between *two* vectors.

Most students successfully identified that two texts with the same frequency vector would be judged as being similar. However, most felt it sufficient to write an original paragraph and claim that, since they had just written it, it could not be by the same author (thereby missing the point that a third person would be evaluating authorship). Two pairs of students gave viable answers: nonsense paragraphs that consisted of nothing but the frequency words.

Sample Module 2: Linear Maps

Whereas many modules were developed by experienced math educators already conversant with the APOS model, this module was developed by a research mathematician in conjunction with the LINE project researchers. He chose linear maps as one of the central themes of his course and proposed a series of problems that students would work on at the start of the study of that material. Line project researchers helped the instructor refine his problems to a set that would be:

- accessible to students in his class prior to formal instruction in this component of the course;
- aligned with the APOS framework for concept development in mathematics; and
- useful to assess student understanding of the material as a guide to instruction.

The instructor participated in the reading seminar online with the other researchers during the summer prior to the course. He then wrote the module reflecting his new understanding of the APOS framework. His initial design goal was to create a series of problems which would scaffold the students' ability to address problems "that require increasing levels of understanding," moving from simple to complex. As we discussed his design, it was not difficult to relate his complexity-based conception to the hierarchy of the APOS framework. The module began with concrete examples concerning linear maps. It then moved on to proofs requiring students to show that a linear map existed between certain vector spaces.

Observations of Student Learning

Some students said that the proof techniques they developed while working on modules carried over into other upper level math courses, such as abstract algebra. Others said the modules helped them think abstractly. Several noted that they liked the concrete examples at the beginning of a module and how they would tie in with the more abstract questions later. They also found the whole-class discussion provided them with significant feedback that comments written on an assignment could not provide. They expressed confidence in their understanding of linear algebra topics and said their knowledge had been enhanced and deepened by the modules.

Students performed quite well on this module, which was the second completed during the semester. As would be expected, the first two questions were correctly answered by most students, any difficulties were with the last two questions. Individual student recollections about the course and this assignment after the end of the semester appeared to match well with their performance during the course. For example, when asked to identify three important concepts in the course a successful student said, “Linear transformations form the basis for a large portion of linear algebra, and since linear algebra overlaps with a lot of other fields (both with math and outside it), and knowing basics like this are important.” When asked whether he could still answer these problems, he responded, “Without my book or another reference, I'm sure I could do 1 and 2, and I could probably start the rest of them. If I were to use references though I'm confident that I could complete all the problems, although it would probably take me longer now than it would have at the end of last semester because I'd have to do some review.” He concluded, “The assignments were generally helpful to learning the material.”

Conclusion

Our final goal was to ensure conceptual alignment between content and pedagogical goals through the use of theories of learning, with an emphasis on APOS, and instructional models. We believe we substantially met this goal, and that it produced noticeable positive effects on student learning: students, prompted to reflect on their learning, volunteered their beliefs that the modules deepened their conceptual understanding.

This project has been more successful than our previous efforts; whereas before we attempted to create this conceptual alignment by offering parallel courses in math content and learning theory, now we focus on collaboration among faculty. The parallel courses didn’t work as well, mainly because students didn’t have time in their schedules for an extra class on learning theory. By concentrating on infusing learning theory into instructional practice, we were able to convey some of these ideas implicitly to students. Collaboration between mathematicians and math educators has been extremely valuable in rethinking instruction in upper-level math classes.

This approach appears to be surprisingly universal: the four institutional settings are quite different—in size, location, student population, and curriculum—but this mode of instruction has led to changes in all of them. Finally, this approach to designing instructional material is flexible. Not every module drew equally from the LINE principles; some emphasize applications, some were technologically enhanced, and some were like carefully-structured labs based on a text.

References

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In Kaput, Schoenfeld, & Dubinsky, (Eds.), *CBMS Issues in Mathematics Education, 6,* 1 - 32.

Hamos, J., Bergin, K., Maki, D., Perez, L., Prival, J., Rainey, D., Rowell, G.,VanderPutten, E. (2009). Opening the classroom door: Professional learning communities in the math and science partnership program. *Science Educator*. *18*(2). 14-24.

Martin, W. O., Loch, S., Cooley, L., Dexter, S., and Vidakovic, D., (2010). ‘Integrating learning theories and application-based modules in teaching linear algebra’. In Brumaldi, Mehrmann, & Schneider (Eds.) *Journal of Linear Algebra and its Applications*, 432. 2089–2099.

Sierpinska, Dreyfus, T., & Hillel. (1999). Evaluation of a teaching design in linear algebra: The case of linear transformations. *Reserches en Didactique des Mathématiques. 19*(1). 7-40.

Vidakovic, D., Cooley, L., Martin, W.O. and Meagher, M. (2008). Making Connections Between the Study of Linear Algebra Content and the Study of Learning Theories. *Proceedings of the Eleventh Annual Conference on Research in Undergraduate Mathematics Education (RUME).* San Diego: RUME - SIG of the Mathematical Association of America (MAA).