

CORRESPONDENCE AND COVARIATION: QUANTITIES CHANGING TOGETHER

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Exponential functions are important topic in school algebra and in higher mathematics, but research on students' thinking suggests that understanding exponential growth remains an instructional challenge. This paper reports the results of a small-scale teaching experiment with students who explored exponential functions in the context of two continuously covarying quantities, height and time. We present two major conceptual paths that occurred in the development of an understanding of exponential growth, the covariation view and the correspondence view, and discuss the influence of each perspective on the growth of students' understanding.

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Introduction

Exponential functions are an important concept in school algebra, representing a significant transition from middle school mathematics to the more complex ideas students encounter in high school. A focus on the conceptual underpinnings of exponential growth has increased in recent years; for instance, the Common Core State Standards in Mathematics (2010) highlight the need to understand exponential functions in terms of one quantity changing at a constant percent rate per unit interval relative to another. Moreover, these ideas also now appear at the middle school level, both in terms of national standards as well as in middle school curricula (e.g., Lappan et al., 2006).

Implementing this vision, however, may prove challenging given students' documented difficulties in understanding exponentiation. Students struggle to transition from linear representations to exponential representations, to identify what makes data exponential, or to explain what a function such as $f(x) = a^x$ means (Alagic & Palenz, 2006; Weber, 2002). Pre-service teachers have not fared much better, struggling to recognize growth as exponential in nature, to connect a repeated-multiplication understanding to the closed-form equation, and to generalize the rules of exponents (Davis, 2009). In general, understanding exponential growth appears to be challenging for both students and teachers (Davis, 2009; Weber, 2002).

These documented challenges suggest a need to better understand how to foster students' learning about exponential growth, particularly in a way that goes beyond merely highlighting the skills students should acquire. This paper reports on the results of a teaching experiment emphasizing exponential growth in the context of two continuously covarying quantities. We introduce two major conceptual paths that occurred in the development of an understanding of exponential growth, the covariation view and the correspondence view, and discuss the influence of each perspective on the growth of students' understanding.

Background and Theoretical Framework

The Rate-of-Change Perspective

A popular approach to function relies on the correspondence perspective (Smith, 2003), in which a function is viewed as the fixed relationship between the members of two sets. Farenga and Ness (2005) offer a typical correspondence definition of function: “One quantity, y , is a function of another, x , if each value of x has a unique value of y associated with it. We write this as $y = f(x)$, where f is the name of the function” (p. 62). This static view of function underlies much of school mathematics, in which a focus on developing correspondence rules between x and y enables students to solve prediction problems, construct and interpret graphs and tables, and determine roots and missing values.

Smith and Confrey (Smith, 2003; Smith & Confrey, 1994) offer an alternative to the correspondence view, which they call the covariation approach to functional thinking. Here one examines a function in terms of a coordinated change of x - and y -values, where moving operationally from y_m to y_{m+1} is coordinated with movement from x_m to x_{m+1} . Confrey and Smith (1995) have found that students’ initial entry into a problem is typically from a covariational perspective; moreover, viewing a function as a representation of the variation of quantities can be a powerful way to promote an understanding of rates of change (Slavit, 1997; Smith & Confrey, 1994).

Castillo-Garsow (2012) describes covariation as the imagining of two quantities changing together; students imagine how one variable changes while imagining changes in the other. Relying on situations that involve quantities that students can manipulate and investigate can foster their abilities to reason flexibly about dynamically changing events (Carlson & Oehrtman, 2005). An approach that leverages covarying quantities may be especially useful in helping students understand exponential growth, as this view is strongly connected to how students think about contexts involving multiplicative relationships (Davis, 2009).

Hypothetical Learning Trajectories

The notion of a *hypothetical learning trajectory* has different meanings among mathematics education researchers. Under Simon’s (1995) original description, a hypothetical learning trajectory consists of “the learning goal, the learning activities, and the thinking and learning in which students might engage” (p. 133). Clements and Samara (2004) elaborate on this definition, describing a learning trajectory as a description of children’s thinking and learning in a specific mathematical domain connected to a conjectured route through a set of tasks designed to support their movement through the progression of levels of thinking. This paper makes use of a hypothetical learning trajectory for exponential functions (reported elsewhere in Authors, 2013) that is an empirically-supported model of the students’ initial concepts and an account of how they changed over time (Confrey et al., 2009). Rather than reporting the trajectory itself, we focus on two conceptual paths represented in the trajectory: the covariation view and the correspondence view. Each path is exemplified through two students’ work as they progressed through the teaching experiment.

Methods

The study was situated at a public middle school and consisted of a 12-day teaching experiment with 3 eighth-grade students (ages 13-14) in which the first author was the teacher-researcher. All students were female. The purpose of the teaching experiment was to support students’ reasoning about exponential growth through exploring continuously covarying quantities. The sessions focused on the relationship between height and time for an exponentially growing plant called the Jactus; students were able to manipulate the plant’s height over time using an interactive computer program called Geogebra. Although this scenario is not realistic,

the context is realizable (Gravemeijer, 1994) in that students could imagine, visualize, and mathematize the relevant quantities. All sessions were videotaped and transcribed.

We assumed that any understanding students might have about exponentiation before entering a teaching experiment would be dependent on an image of repeated multiplication. Building on that conception, our primary goal of instruction was to foster students' understanding of the following set of concepts for an exponential function $y = a \cdot b^x$:

1. The period of time x for the y -value to increase by the growth factor b is constant, regardless of the value of a or b .
2. There is a constant ratio change in y -values for each constant additive change in corresponding x -values.
3. The ratio of the change in y , which can be expressed as $\frac{y_2}{y_1} = b^{x_2-x_1}$, is always the same

for any same Δx and the value of $f(x + \Delta x)/f(x)$ is dependent on Δx .

Data analysis relied on retrospective analysis techniques (Simon et. al, 2010) to characterize students' changing conceptions throughout the course of the teaching experiment. Project team members developed preliminary codes for concepts on the trajectory based on students' talk, gestures, actions, and task responses as evidence of understanding at different stages. The first round of analysis yielded an initial set of codes, which then guided subsequent rounds of analysis in which the project team met as a group to refine and adjust the codes in relationship to one another; this iterative process continued until no new codes emerged. Once coding was complete we chose 20% of the data corpus for independent coding, which yielded an inter-rater reliability rate of 92%. For the purposes of this report, we focus on two students, Udit and Jill, as their work provided a way to contrast the influence of a correspondence view versus a coordination view on subsequent concept development.

Results

The progression of the students' understanding of exponential growth occurred in three major stages, which we call *pre-functional reasoning*, *covariation reasoning*, and *correspondence reasoning*. The learning trajectory is depicted visually in Figure 1 for brevity's sake (for a complete report of all of the codes, associated tasks, and data examples see Authors, 2013.) Although pre-functional reasoning preceded the development of the covariation and correspondence views, the latter two ways of thinking did not occur in a sequential nature. While all of the students first developed an early, rudimentary coordination between height and time, they then diverged, some focusing on a more sophisticated covariation perspective, some gravitating towards a correspondence view, and others simultaneously developing both. An emergent goal of the teaching experiment was to support students shifting flexibly between these perspectives as needed.

In the following sections we report on two students, Jill and Udit, as a way to exemplify a focus on correspondence versus a focus on covariation. All of the students readily identified repeated multiplication as the mechanism determining how the Jactus grew, but did not initially connect repeated multiplication in height to the unit of time. In an attempt to encourage coordination of the plant's height with the number of weeks it had been growing, the teacher-researcher introduced a task requiring students to draw the plant's height after 1 week and after 3 weeks if it doubled every week. For example, Jill's drawing indicated the beginning of her coordination of two quantities; both are present in her picture (Figure 2), and Jill explained that the plant's height for Week 4 would be "double the last week; like Week 4 would be 16 inches."

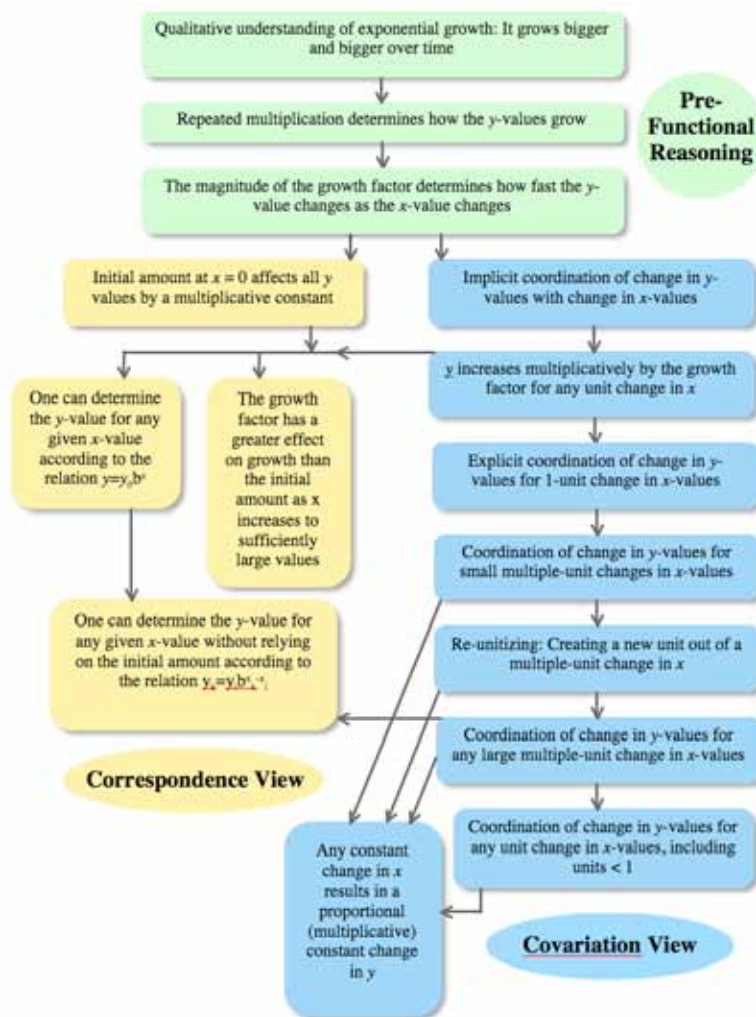


Figure 1: Graphical Representation of Exponential Functions Trajectory

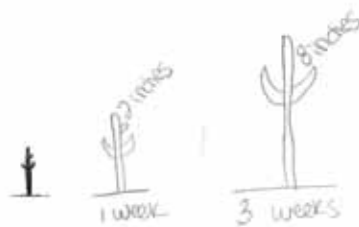


Figure 2: Jill’s Drawing of the Growing Jactus

Jill: A Focus on Correspondence

Once Jill determined that the growth factor represented the multiplicative change in height per week, she shifted to a reliance on expressing this relationship algebraically. Jill first developed an algebraic relationship when provided with a data table in which she had to

Martinez, M. & Castro Superfine, A (Eds.). (2013). *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Chicago, IL: University of Illinois at Chicago.

determine missing height values (Figure 3):

Weeks	Inches
0	3
1	6
2	12
3	24
5	96
8	768
10	3072
13	24576
x	?

Handwritten notes: $3 \cdot 2^x$, starting number, goes up by 2

Figure 3: Jill’s Strategy for Determining Missing Values in a Data Table

Jill appeared to determine the expression $3 \cdot 2^x$ by guessing and checking to find the value of 3. She explained, “You times this number by 2 [justifying 2^x], and then I don’t know why the 3 works.” Through comparing different doubling plants with various starting heights, Jill began to develop an understanding that in order to determine the correct height at a given week x , she had to multiply 2^x by the initial height, which she called the “starting number”: “Well, the starting number of the week 0 is 3, and then do 2 to the x because it’s increasing by times 2 each week.” In this manner Jill developed a conception of the equation $y = ab^x$ as the “starting number” times “how much it goes up by”, where “goes up by” refers to multiplicative growth.

Jill’s understanding of the role of the initial height was as the value from which one begins the repeated multiplication process. Once Jill could reliably produce a relation between height and time, she made use of this correspondence rule to determine the growth factor. For instance, when Jill encountered a scenario with only three data points: (0; 1), (20; 1,048,576), and (25; 33,554,432), she explained, “I knew that it would be 1 times something to the x power so I started with 5 [for the missing growth factor] and that was just way too big, and so then I did 3 and it didn’t work still, so I did 2 and then it worked and then I plugged in 20 [to check her answer]. So I did 1 times 2 to the 20th power and I got that [pointing to 1,048,576].” Jill’s reliance on the correspondence rule shifted her attention to guess-and-check methods to find the missing values that filled in the “slots” in $y = ab^x$. Although this method is cumbersome, it allows one to avoid taking a ratio to determine the growth factor (for instance, determining the growth factor by taking the ratio of 33,554,432 to 1,048,576, which is 32, and solving the equation $x^5 = 32$.)

Jill’s understanding of the correspondence rule was powerful in that it enabled her to determine missing values; however, it also constrained her thinking. For instance, Jill had to rely on the development of an equation in order to convince herself of the correctness of a growth factor, even when it would be easier to do so by more direct methods. Given three data points for a plant with an unknown growth factor [(8; 19,660.8), (15; 322,122,547.2), and (18; 20,615,843,020.8)], Jill took the plant’s height at 8 weeks and multiplied it by different growth factors seven times until she hit upon the correct height value at 15 weeks. Although Jill determined the growth factor was 4 in this manner, she was not convinced of its correctness absent a correspondence rule. Jill therefore divided 19,660.8 by 4 eight times in order to determine the initial height of 0.3 inches. It was only when Jill wrote the equation 0.3×4^x that

she could check data points and gain confidence that the growth factor was indeed 4.

The correspondence perspective was so prevalent for Jill that she typically thought in terms of static height and time values rather than imagining how the plant could grow over time; this may be why Jill was unconvinced of her answer of 4 in the problem above. In another example, given a plant that grew 8 times as tall every three weeks, Jill was unable to determine, in absence of an equation, how the plant grew each week. Jill created a sample table of values depicting the plant's height as 1 inch at 0 weeks, 2 inches at 1 week, 4 inches at 2 weeks, and 8 inches at 3 weeks. Despite the creation of this table, Jill was at a loss to determine the plant's growth factor, explaining, "I tried to divide 8 by 3 but that didn't really do anything." Jill's restriction to the correspondence view may have hampered her ability to see the growth factor; she may have viewed each entry in the table, such as (1,2), as a static height value, e.g. 2 inches tall at 1 week. A covariation perspective, in contrast, could enable one to also view the point (1, 2) in relationship to the prior point, (0, 1), and see that the plant has doubled in height from 1 inch to 2 inches in the span of 1 week.

Uditi: A Focus on Covariation

Like Jill, Uditi was able to describe the plant's growth in Figure 2 in a covariational manner: she explained, "In one week it's going to be 1 inch and in 2 weeks it's going to be 2 inches, then in 3 weeks it's going to be 4 and in 4 weeks it's going to be 8." When Uditi examined tables of data with jumps in weeks, she initially had to imagine the missing weeks in place. For instance, for Figure 4, Uditi imagined the missing Week 11 in the table: "For the 12 weeks I did times 4 for the answer [to week 11] then I did times 4 for that [to get the height at week 12]." In this manner, Uditi began to coordinate how the height grew with how the weeks grew, keeping track of both and mentally filling in the gaps when necessary.

Eventually Uditi began to formally coordinate the multiplicative growth in inches with the additive growth in weeks. For instance, given a table with a height of 956,593.8 inches at Week 14 and 8,609,344.2 inches at Week 16, Uditi marked a difference of 2 between the successive weeks and a ratio of "×9" between the successive inches. She then described her process: "I divided 8,609,344.2 by 956,593.8 and I got 9. Then I tried to figure out what number times itself = 9 because the difference between 14 and 16 is 2." In this manner she concluded that the growth factor must be 3, but could not yet generalize her reasoning to any multi-week gap.

Weeks	Inches
0	0.1
2	1.6
5	102.4
9	26,214.4
10	104857.6
12	167721.6

Figure 4: Uditi's table of values

In order to encourage this generalization, we introduced tasks with larger gaps between the weeks, such as a plant with an unknown growth factor that is 256 inches at 4 weeks and 1,073,741,824 inches at 15 weeks (Figure 5). The 11-week gap was sufficiently large to discourage Uditi from relying on a mental image of repeated multiplication. Instead, Uditi took the ratio of the height values, which is 4,194,304, and wrote " $___^{11} = 4,194,304$ ". She was then able to determine the missing growth factor that satisfied the equation was 4.

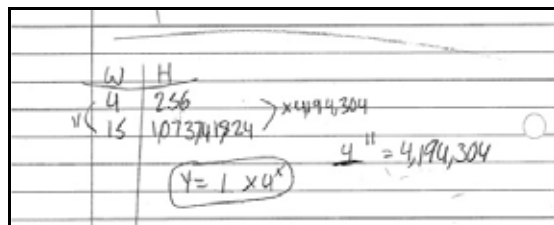


Figure 5: Uditi's strategy for determining the growth factor

In generalizing this approach, Uditi began to coordinate height and time values when $\Delta x < 1$. For instance, given two height values for Week 2.3 and Week 2.4, Uditi could divide them to find the multiplicative ratio, 1.12, and then write " $___^{0.1} = 1.12$ " in order to determine the growth factor. Tasks asking students to predict how the same plant would grow for different increases in time also fostered the covariation perspective for cases in which $\Delta x < 1$. One such task introduced a plant that tripled each week, asking students to consider how much larger it would grow in 1 day. Uditi responded to the question with " $3^{.14} = 1.17$ ", explaining, "I divided 1 week into 7 parts, which represents 1 day each and it's .14 of a week." In this manner Uditi was able to make sense of an expression with non-whole number exponent, viewing $3^{.14}$ not just as a static height value, as Jill did, but as an expression of the *growth* in height for 1 day.

Discussion

The expression b^x can both represent a static value (for instance, the height of a plant at time x) and a measure of growth (for instance, how much taller the plant will grow in x amount of time). A correspondence perspective encourages the first conception by emphasizing the direct relationship between height y and time x in the equation $y = ab^x$. Both Jill and Uditi easily understood the expression b^x as a specific height value at a given time. Both students also experienced difficulty in transitioning to an understanding of b^x as a representation of growth in height. We expected this difficulty with Jill, given her focus on correspondence relationships. However, Uditi also found this transition challenging despite her continued emphasis on coordinating multiplicative growth in height with additive growth in time. Uditi's strong early reliance on an image of repeated multiplication may have contributed to this challenge: When conceiving of b^x as a representation of b multiplied by itself x times, it is difficult to think about multiplying a number by itself a fractional amount of times. It is easier to conceive of the x in b^x as a "slot" in which one places the time value, as Jill did.

Uditi's ability to coordinate the ratio of height values with the additive difference in time values played a significant role in supporting the development of algebraic representations. Ultimately, it was easier for Uditi to shift flexibly to a correspondence view, as needed, than it was for Jill to shift to a covariation view. Moreover, Uditi's understanding of the correspondence rules reflected her covariation focus. She understood that when finding the height value at a specific time ($y = ab^{4x}$), she was multiplying the height at week 0 (a) by the multiplicative growth for that time span (b^{4x}). In other words, Uditi understood that in order to find the height of the plant, she needed to multiply by the growth factor for every time unit passed. Her reliance on covariation reasoning also enabled her to develop a more sophisticated understanding of the correspondence view in which she no longer needed to start the process from week 0. Instead, she was able to find the height value at a specific time based on the height value at any given week (different than week 0).

In general, a rudimentary level of coordination of growth in height as “doubling each time” with growth in weeks preceded every student’s ability to develop correspondence rules of the form $y = f(x)$, which reflects Smith and Confrey’s (1994) assertion that students typically approach functional relationships from a covariational perspective first. Our preliminary findings suggest that maintaining an emphasis on covariation may actually assist students’ eventual construction and understanding of correspondence rules; therefore, a covariational focus may be the most efficacious in supporting students’ understanding of exponential growth.

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