VISUALLY GROUPING OPERANDS: PERCEPTUAL FACTORS INFLUENCE ARITHMETIC PERFORMANCE

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Perceptual characteristics of mathematical equations may influence solvers' problem solving. For example, in a study of equations involving addition and multiplication, Landy and Goldstone (2010) showed that participants tended to perform narrowly spaced operations first, suggesting that spacing affects how symbols are grouped for problem solving. Building on this past work, we examined whether perceptual factors affect participants' interpretations of the minus sign. In an experiment with undergraduates, we manipulated the spacing of the operands and the position of the minus sign relative to its neighboring operands. Both the operands' spacing and the position of the minus sign affected performance. These results hold implications for the processing of symbolic representations and for mathematics education.

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In the abstract symbol system of mathematics, individual numerical and operational symbols are combined into more complex mathematical expressions. These combinations take place in a two-dimensional representational system, in which spatial features may be relevant or irrelevant to formal mathematics. For example, a relevant spatial feature would be the ordering of the digits and operators in a given mathematical equation—because mathematically correct solutions to multi-operation arithmetic problems are reached by following mathematical rules, such as the order of operations. Similarly, the size and spatial location of a superscripted digit to indicate an exponent differentiates its mathematical meaning from the digit used as the base. In contrast to these examples, there is no difference in the formal mathematics of an expression when the horizontal spacing of the symbols is varied; correct application of the order of operations rule should yield the correct answer, regardless of how the equation is horizontally spaced.

However, some recent evidence suggests that, despite its formal irrelevance to mathematical meaning, horizontal spacing affects problem solvers' solutions. Landy and Goldstone (2007a) found that the amount of space surrounding operands influenced participants' judgments of the equality of two mathematical expressions. For example, undergraduate participants were more likely to agree with the statement "Is a + b * c + d = b + a * c + d necessarily true?" when the multiplication operations were more widely spaced than the addition operations (i.e., a + b   *   c + d = b + a   *   d + c), compared to when they were equally spaced. Participants made more errors when spacing was inconsistent with the mathematically correct order of operations (as discussed in Landy & Goldstone, 2007b).

Effects of spacing also are present when people construct mathematical expressions. Evidence from problem transcriptions show that participants implicitly follow spatial patterns when converting equations from word form (e.g., three plus five times two) to number form (e.g., 3 + 5 x 2). Participants spaced multiplication signs more narrowly than plus signs, signifying the precedence of multiplication (Landy & Goldstone, 2007b).

Effects of spacing on performance are also apparent in the time it takes for participants to solve problems (Landy & Goldstone, 2010). Participants solved equation structures of the form "a + b x c" or "a x b + c" faster and more accurately when the multiplication operation was spaced more narrowly than the addition operation (e.g., a + bxc or axb + c, as compared to a+b x c or a x b+c). Participants displayed higher accuracy on problems with spacing that was consistent with the order of operations rules, and they also processed those equations more quickly.

These findings indicate that, irrespective of its formal irrelevance, the horizontal spacing and proximity of operands and operations can influence people's interpretation and understanding of mathematical expressions. Moreover, it appears that the relationship between horizontal spacing and arithmetic performance is bidirectional. People's knowledge about the order of operations can influence the spacing they use when writing expressions, and spacing can affect the order in which they perform operations when evaluating or solving problems. Furthermore, these findings may have important implications for educational practice. Instructors may be able to use these perceptual features to their advantage in classroom instruction.

In light of these previous findings, the minus sign (i.e., "–") is a particularly interesting case, because it is used not only to represent subtraction, but also to invert the sign of a number. Consequently, the minus sign is sometimes treated differently from other operation symbols; for example, it is "carried" with the associated operands in algebraic manipulations. This can present challenges for students learning algebra (Cangelosi, Madrid, Cooper, Olson, & Hartter, 2013; Demby, 1997; Vlassis, 2004). In addition, as a mathematical operation, subtraction is more difficult than addition (e.g., Das, LeFevre, & Penner-Wilger, 2010). In this study, we investigate whether formally irrelevant perceptual features can influence how the minus sign is interpreted. If perceptual features influence subtraction, as they do addition and multiplication, our research findings may be important for understanding students' difficulties at the transition between arithmetic and algebra.

In the present research, we investigated the effects of perceptual grouping of operands on participants' interpretations of the minus sign. Participants were presented with multi-operation expressions that involved subtraction. The equations were in the format of "a – b + c x d". We tested whether undergraduates' arithmetic accuracy would be enhanced by supportive yet formally irrelevant perceptual cues consistent with order of operations rules or hindered by misleading, formally irrelevant perceptual cues inconsistent with orders of operations rules.

Building on Landy and Goldstone's previous work (2007a, 2007b, 2010), we varied two aspects of spacing: the closeness of *operand spacing* and the lateral *minus sign position*. Before presenting our experimental method, we describe each of these variations and how they might affect performance. In both cases, the variations we employed were subtle, consisting of only one or two spaces in a normal-sized font.

First, we varied the *operand spacing*, specifically for the operators that followed the minus sign. To do so, we removed the spaces between operands and operations for the last three terms, similar to Landy and Goldstone's (2007a) manipulation. The closer proximity of the symbols may lead solvers to perceptually group these terms, as suggested by the Gestalt principles of visual perception (e.g., Werthmeimer, 1923/1938). Evidence from past research (e.g., Landy & Goldstone, 2010) suggests that visually grouping these symbols should give them precedence in problem solving steps. That is, the narrower spacing of these symbols (e.g., 25 – 3+2x5) could create a perceptual group which might lead participants to construe "3+2x5" as the subtrahend, and mistakenly apply the subtraction operation to this entire quantity. This would lead to the error described in Table 1, which we term a "target error". In erring this way, participants incorrectly evaluate the scope of the minus sign, due to the perceptual grouping.

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Second, we varied the *minus sign position,* again based on the Gestalt principle of proximity (Wertheimer, 1923/1938). We horizontally shifted the minus sign (see Table 2 for a visual display) to create different perceptual groups. For instance, in the equation "25 – 3 + 2 x 5 = ____", if the minus sign were shifted slightly to the left, (25 – 3 + 2 x 5 = ____), solvers should be more likely to group the remaining symbols "3 + 2 x 5". This, too, may lead participants to produce the target error, because the minus sign may be applied to the entire quantity of "3 + 2 x 5" because of the perceived group. In contrast, if the minus sign were shifted to the right, solvers should be more likely to group the operator with the subsequent subtrahend of 3. Thus, manipulating the lateral position of the minus sign has the potential to affect the quantity participants treat as the subtrahend.

Table 1: Incorrect Solution Strategy Based on Incorrect Perceptual Grouping

Problem	25 – 3 + 2 x 5 = ____	correct answer = 32
Step 1	multiply: 25 – 3 + 10 = ____	correctly following order of operations
Step 2	add: 25 – 13 = ____	Demonstration of a "target error" with incorrect order of operations, potentially based on perceptual grouping of all symbols to the right of the minus sign
Step 3	subtract: = 12	

Although we could have tested many other perceptual features, we chose these two manipulations because of their implications for how the subtrahend is determined. The *minus sign position* manipulation directly alters the proximity of the minus sign to other elements in the equation; this manipulation relies on the proximity of the operator to its operands to generate supportive or misleading perceptual groups. The *operand spacing* manipulation more subtly affects the how the minus sign is drawn into the subtraction operation; the closeness of the latter terms may again prompt a perceptual grouping that influences a solver's interpretation of the scope of the minus sign by affecting the perceptual group that is treated as a subtrahend.

In sum, we hypothesized that perceptual cues inconsistent with the order of operations rules would encourage solvers to incorrectly group operations together and lead to errors in problem solving. Both the mathematically irrelevant spatial features of minus sign position and spacing of the remaining operands could potentially affect how the scope of the minus sign is interpreted. Despite their formal irrelevance, we expected these manipulations to affect participants' solutions to arithmetic problems involving the minus sign.

Methods

Participants

Undergraduate students ($N = 92$) in introductory psychology at a large Midwestern university participated in exchange for extra credit. Sixteen additional participants were excluded due to a photocopying malfunction leading to faulty stimuli.

Design and Materials

Participants solved arithmetic problems with manipulated perceptual features. In the target problems, subtraction was the first operation presented the equation, followed by addition, and finally multiplication (e.g., 25 – 3 + 2 x 5).

A 2 (*operand spacing*: evenly spaced or closely spaced) x 3 (*minus sign position*: left shift, no shift, or right shift) between-subjects design was used, yielding a total of 6 conditions. Table 2 displays the outcomes of these manipulations. In the evenly spaced problems with no shift, there were two spaces between each operand and the adjacent operator and two spaces around the equal sign. The left-shift condition was created by reducing the space between the first operand and the minus sign by one space and adding a space between the minus sign and the subsequent operand. The opposite was done for a right shift. For the closely-spaced condition , all the spaces after the second operand were removed, which resulted in no spaces between the last two operators and their three surrounding operands, as seen in the first row of Table 2.

Table 2: Example Target Problem with Shift and Grouping Manipulations

	Minus Sign Position		
Operand Spacing	**Left Shift**	**No Shift**	**Right Shift**
Evenly Spaced	25 – 3 + 2 x 5 = ____	25 – 3 + 2 x 5 = ____	25 –3 + 2 x 5 = ____
Closely Spaced	25 – 3+2x5 = ____	25 – 3+2x5 = ____	25 –3+2x5 = ____

Participants received ten target problems and eight control problems of the form "a + b – c x d". The control problems did not afford the target error that the target problems did. The operands in the equations were randomly generated, with the constraint that both the operands and solutions were non-zero positive integers. The problems in each condition were presented in a fixed order and the same operands were used for any given problem across conditions.

On each of the 8.5" x 5.5" (wide) pages of the packet, there were two math problems, separated by a fill-in-the-blank vocabulary question (as seen in standardized tests such as the SAT) as a filler. Size-12 Calibri font was used for all the questions, and the questions were numbered at the left margin. The packets were constructed such that the pages with the target math problems alternated with pages with the control math questions.

Procedure

Participants were tested in groups of up to five students at a time; each completed his or her packet of problems individually. Each participant was given a question packet with the math problems and vocabulary filler questions. Participants had up to thirty minutes to complete the question packet. Once participants finished the problem set, they were asked to provide information regarding their math and reading backgrounds and other relevant demographic information such as year in school and major.

Results

We scored whether each solution on each target problem reflected the target error or not, that is, whether the participant used a mathematically incorrect but perceptually-based group to solve the problem. Thus, careless arithmetic errors (e.g., 3 + 3 = 9) and errors that involved solving from left to right without consideration of order of operations (e.g., 25 – 10 + 3 x 2 = 36), while mathematically incorrect, were coded as No Target Error.

We analyzed the existence of target errors with a mixed effects logistic regression model using the glmmadmb package (Skaug, Fournier, & Nielsen, 2012) in R. Our two factors, minus

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sign position and operand spacing, were treated as fixed effects that were allowed to interact, and participant was a random effect.

Figure 2 presents the mean proportion of trials with target errors in each condition. When the operations that followed the minus sign were closely spaced, participants in the left shift condition produced significantly more target errors than participants in the right shift condition, $p = .028$. The left-shift condition did not yield significantly more target errors than the no-shift condition, although the results trend in this direction. Additionally, within the closely-spaced condition, participants in the no-shift condition produced marginally more target errors than participants in the right-shift condition, $p = .059$. However, in the evenly-spaced condition, there was no difference in the proportion of target errors among the left-shift, no-shift, or right-shift conditions.

Focusing on the effects of the position manipulation, when the minus sign was shifted to the left, participants in the closely-spaced condition produced more target errors than participants in the evenly-spaced condition, $p = .026$. Close spacing also led to significantly more target errors than equal spacing when there was no shift, $p = .038$. However, when the equal sign was shifted right, there was no difference between the closely- and evenly-spaced conditions, $p > .05$.

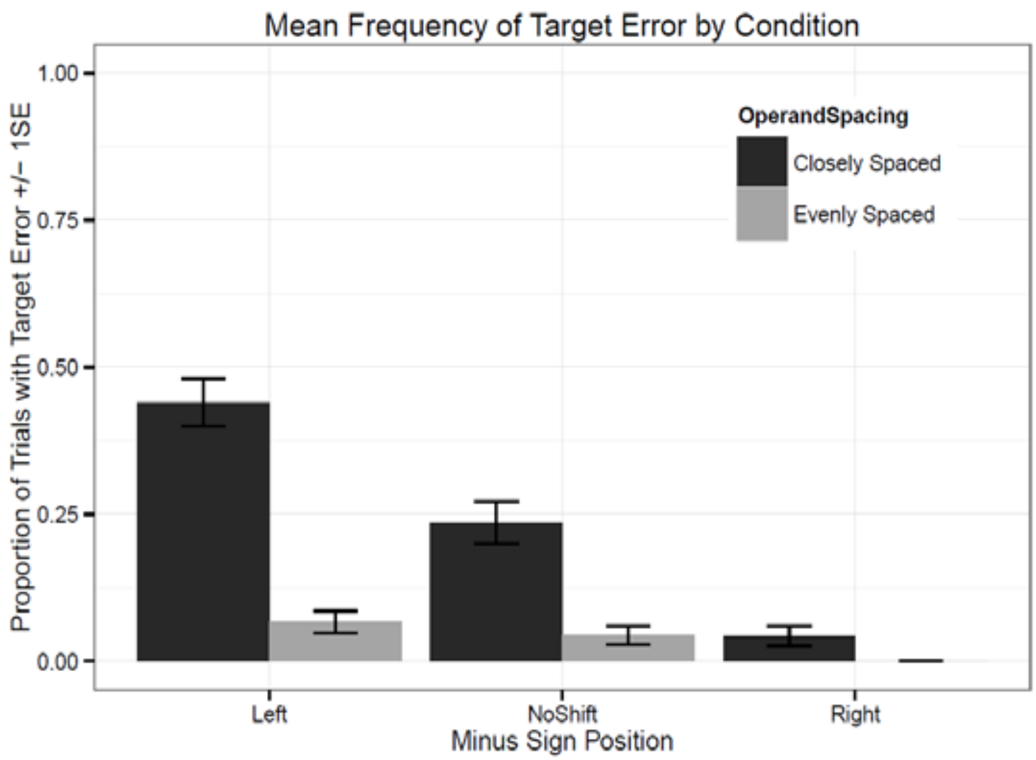

Figure 1: Proportion of Trials with Target Errors by Condition

In summary, target errors were most frequent when the minus sign was shifted left and the operations that followed the minus sign were closely spaced. Closely-spaced symbols were associated with more target errors than evenly-spaced symbols, for both the left shift and no shift conditions. Overall, the proportion of target errors within the closely-spaced condition was affected by the position of the minus sign. On the other hand, target errors were very low when the operands were evenly spaced, and the proportion of target errors within the evenly-spaced conditions was not affected by the position of the minus sign.

Discussion

This study investigated whether spatial properties of equations would encourage participants to misevaluate the scope of the minus sign. We found that shifting the minus sign to the left and closely spacing the operations that followed the subtraction operation increased the frequency of the target error. This means that participants used the perceptually-based groupings instead of the mathematically-correct grouping, as defined by order of operations rules. Thus, when the manipulations afforded perceptual groups, participants were more likely to perform the perceptually grouped operations first, despite the inconsistency with formal mathematics. The variations we implemented in the position of the minus sign and spacing of the operations did not alter the inherent meaning of the equations. However, the participants had more difficulty accurately processing equations with perceptual features that were inconsistent with the order of operation rules. These findings extend past research by Landy and Goldstone (2010), and they indicate that multiple ways of manipulating perceptual grouping can influence participants' interpretations of mathematical equations and guide their problem solving.

Equations with the final three operands spaced closely elicited more target errors, except when the minus sign was shifted right (e.g., 25   – 3+2x5 = ____). In the right-shift condition, participants seem to have incorporated the minus sign as part of the subtrahend, and this may have protected them from making the target error. This suggests that participants in the right-shift condition were either more likely to perform the multiplication and then go back to the beginning of the equation *or* to realize that the minus sign could be applied to the subsequent operand, making it negative. In contrast, participants in the left-shift and closely-spaced condition, in which the target error was most prevalent, tended to ignore the minus sign (perhaps because it was spatially more distant) and to incorrectly perform the subsequent operation without the minus sign "riding along" (i.e., to calculate "3 + 10" instead of "–3 + 10" in the intermediate step).

The increased frequency of target errors in the left shift condition compared to the right shift condition suggests that perceptual factors may influence whether solvers activate the notion of negative numbers. The proximity of the minus sign to the subsequent operands in the right shift condition may have allowed participants to "attach" the minus sign to the subsequent perceptual group (e.g., in Table 1, "–3+2x5"), so that they viewed the subsequent operand (i.e., 3) as a negative integer. The more distant minus sign in the left shift condition may not have afforded this connection. This possibility is compatible with evidence from Vlassis (2004) suggesting that at the transition from arithmetic to algebra, students expand their understanding from natural numbers to integers, which may incorporate the minus sign to indicate that an integer is negative. However, our findings suggest that, when there is conflicting perceptual information, undergraduates may not automatically activate the notion of subtraction as adding a negative number.

There are many potential directions for future research in this area. One valuable next step would be to conduct a detailed analysis of participants' problem-solving steps, in order to better understand the processes through which perceptual spacing affected performance on these equations. It is possible that participants in the closely-spaced condition were more likely to start their solution process at the end of the expression (i.e., focusing on the grouped operands first).

This would further bolster the claim that perceptually grouped operations are likely to be performed first.

A second important future direction would be to directly investigate the relative prevalence of spatial and semantic information in problem solving. Our results suggest that spatial information may be processed *before* evaluating the semantic values of symbols. We suggest this given the evidence that participants used the perceptual groups to determine the subtrahend, rather than using the scope of the minus sign as defined by the order of operation rules. However, further research that directly investigates the integration of spatial and semantic information is needed.

This work could also be extended by investigating different types of problems, working with younger students who have less expertise at these operations, and considering the effects of math ability and attitudes on susceptibility to the making the target error. More generally, a deeper understanding of the role of the spatial factors is needed for a complete account of students' acquisition of computational and algebraic skills.

Finally, we believe that these data have important implications for educational practices. Perceptual features of mathematical expressions are a relatively little-studied area in mathematics education. However, perceptual features do affect performance, as we have shown here (see also Landy & Goldstone, 2007a, 2010). It could be beneficial to leverage the effects of spatial features on mathematical processing in instructional contexts. For example, spatial manipulations could be used to support students' learning of the order of operations rules, and spatial support could later be faded as students gain proficiency. Similarly, spatial cues based on proximity or other perceptual features (see Wertheimer, 1923/1938) could be implemented to reduce students' misinterpretations of the scope of the minus sign. Although the research here focused on proximity cues that afforded perceptual groups in arithmetic equations, mathematics in general uses a symbol system that incorporates spatial features in its representation. Overall, our findings suggest the need for a deeper consideration of the spatial characteristics of symbolic expressions used in mathematics instruction.

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