EQUIVALENCE AND EQUATION SOLVING WITH MULTIPLE TOOLS: TOWARD AN INSTRUCTIONAL THEORY

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This research study was specifically concerned with the development, testing, and revision to an instructional theory for studying the mathematical concepts of equivalence and equation solving with multiple representations and multiple tools. Following a design research approach, a collaborative teaching experiment was conducted with a ninth-grade algebra teacher in which instruction was guided by a specifically designed sequence of tasks, techniques using paper-and-pencil and computer algebra systems (CAS), and theory on a hypothesized progression of learning. Retrospective analyses of data informed revisions to a resulting progression of learning and activity sequence that are being tested with pre-service secondary mathematics teachers.

Keywords: Algebra and Algebraic Thinking, Technology, Learning Trajectories (or Progressions), Design Experiments

Background and Purpose

Mathematical equivalence is a central topic in mathematics, and the domain of school algebra specifically (Chazan & Yerushalmy, 2003). Current expectations for school mathematics include justifying the equation solving process by reasoning about equivalence of equations (Common Core State Standards Initiative [CCSSI], 2010). Empirical studies on students' understanding of the equal sign have shown that students hold a diversity of understandings from operational to relational (Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012), and that these understandings matter for students' abilities to solve equations (Knuth, Stephens, McNeil, & Alibali, 2006).

A specific motivation for this study was to link research and practice on the use of computer algebra systems (CAS) to support students' development of representational fluency in solving equations. CAS are a representational toolkit that facilitate the creation of, manipulation of, and movement between symbolic, graphic, numeric, and verbal representation types. For this study, representational fluency was defined as the ability to create, interpret, transpose within, translate between, and connect multiple representations in doing and communicating about mathematics. This construct helps to characterize students' conceptual understanding of mathematics because it deals with cognitive connections across representations of mathematical objects.

Some of the specific gaps in the literature that this study sought to address include the articulation of aspects of CAS-equipped classroom learning environments. For instance, more research is needed to specify the roles of representations and a balance between CAS and paper-and-pencil tools (e.g., Kieran & Saldanha, 2008). Additionally, a focus on the role of the equal sign may shed light on student thinking and understanding with respect to equivalence and equation solving (Knuth et al., 2006). This is also tied to the use of language in learning the nuances and relationships among expressions, equations, equivalence, and solutions (Kieran & Drijvers, 2006). True to the goal of linking research and practice, designing instruction based on students' learning processes pushes the field forward in defining a theory of teaching (e.g., Sztajn, Confrey, Wilson, & Edgington, 2012). On the topic of understanding of equivalence and equation solving, research studies on student learning processes that undergird a meaningful

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instructional sequence are emerging (Kieran & Drijvers, 2006; Kieran & Sfard, 1999; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2010).

This report focuses on a research question that guided a component of research conducted by Fonger (2012): *What means of support seem to facilitate students' development of representational fluency in a combined CAS and paper-and-pencil environment?* To address this question, following Cobb (2003), the means of support were organized around four main aspects: (1) the instructional tasks and activity sequence, (2) the tools students would use, (3) the activity structure of the classroom, and (4) classroom expectations. Inspired by Kieran and Drijvers (2006), the focus of this report is on the tested and revised sequence of tasks, techniques, and theory. The tasks specify the mathematics that students do and the techniques are the ways in which CAS and/or paper-and-pencil are used to accomplish some mathematical goal. The theory is an empirically based conceptual progression of expected tendencies that students will encounter as they formalize their understandings through engagement with tasks, tools, techniques, and other interactions (cf. Kieran & Drijvers, 2006; Sztajn et al., 2012).

Research Design and Theoretical Frameworks

A design research approach (Gravemeijer & Cobb, 2006) was followed in order to effectively design for, test, and revise an empirically based instructional theory. The three phases of this research were: preparation for the experiment, conduct of a teaching experiment and ongoing analyses, and retrospective analyses. During the first phase of the research a conjectured instructional theory was posited based on a review of relevant literature, briefly summarized above and elaborated by Fonger (2012). Conjectured and revised elements of the instructional theory are elaborated in the next section.

The second phase of the research involved a teaching experiment conducted in collaboration with a classroom teacher (Cobb, 2000). The teacher had four years of experience teaching courses in algebra and geometry and had used non-CAS graphing calculators in her instruction. The setting of the research was an algebra classroom at a large urban public high school. During the five-week teaching experiment, the teacher taught all lessons and the researcher served as a participant observer in the classroom. Consistent with the research design and the researcher's epistemological foundations, an interpretive lens on classroom interactions guided the ongoing analyses of classroom activity; the classroom practices and students' mathematical activity and cognition were seen to co-evolve over time (Cobb & Yackel, 1996).

The process of carrying out and testing the instructional sequence involved three aspects that occurred on a daily basis. First, daily cycles of classroom implementation were guided by hypothetical learning trajectories (Simon, 1995) in which learning goals, learning activities, and hypotheses of students' learning process were tested. The second component was reflective in nature with the directive to link research and practice. The teacher and researcher engaged in thought experiments or directed reflections every day after class for 45-90 minutes with student work, lesson notes, and task-technique-theory frameworks to explore the questions "is instruction meeting the set-out goals?" and "how should we improve the next lesson to account for student understandings demonstrated in the most recent lesson?" On Fridays we met for an hour or more to address the questions "how did the weekly teaching sequence support or differ from the conjectured instructional theory?" and "what revisions should be made for next week's plan?" After each thought experiment, the researcher identified critical moments that occurred during the classroom episodes and summarized those in daily and weekly summary files. Critical moments were identified as segments of a teaching episode that seemed to well-support or contradict the proposed learning goals and means of support being tested.

The third phase of the instructional experiment included creating new hypothetical learning trajectories for the next class session that took into account the daily instruction. Figure 1 illustrates the overarching instructional theory and the reflexive relation between theory and practice, which guided ongoing analysis and experimentation.

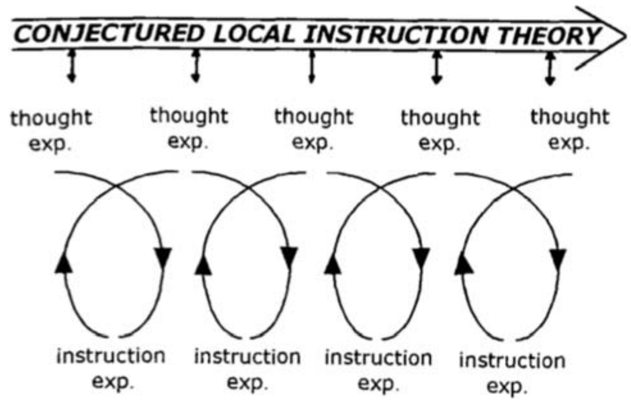

Figure 1: Daily cycles of experimentation were guided by an overarching instructional theory (Gravemeijer & Cobb, 2006).

The final phase of the research involved retrospective analyses of all data. This included: pre- and post-test data from select students of the classroom, initial and final interviews with select students (discussed by Fonger, 2012), teacher and student classwork, classroom and individual student video, observational field notes, daily and weekly class summaries and debriefing notes from collaborative and individual thought experiments about daily instructional episodes, and weekly debriefing session notes. Both Studiocode and HyperRESEARCH tools supported management and analysis of all data.

The data analysis method during the retrospective analysis stage resembled what Gravemeijer and Cobb (2006) described as a constant comparative method (Glaser & Strauss, 1967). In this process, conjectures about the instructional theory that had been identified during ongoing analyses were later confirmed or refuted based on evidence from a given classroom episode; these conjectures were then tested again against the subsequent episode. This process of confirming and refuting conjectures was repeated until all teaching episodes and critical moments were analyzed in chronological order. Critical moments were coded according to the main aspects of the instructional theory: Activity Sequence, Activity Structure, Learning Progression, and Classroom Expectations. As code names within each category were refined (e.g., clarified description, new code name, etc.), all data in that category were re-coded according to the updated code categories. By design, all instructional theory components and descriptions were revised throughout the ongoing and retrospective analyses.

Results: An Emerging Instructional Theory

An instructional theory involves two main aspects: learning processes guided by learning goals, and means of support for tasks, tools, classroom culture, and the role of the teacher (Gravemeijer & Cobb, 2006). The learning goals that defined the direction of the instructional design were to: (a) develop representational fluency with linear expressions and equations, (b) understand the equal sign as an equivalence relation, and (c) solve linear equations as a process of reasoning about equivalent equations. These three goals guided the sequence of learning activities or tasks. CAS and paper-and-pencil techniques were specified in the activity design as a way to accomplish these goals. These techniques were summarized as an activity structure that was grounded in literature on the coordination of tool use (e.g., Kieran & Saldanha, 2008) and a multi-representational lens on doing and communicating mathematics (e.g., Kieran & Sfard, 1999). The theoretical component was intertwined with the tasks and techniques in support of the learning goals.

Tasks

To describe the overall sequence of activities, both the enacted and revised versions of the sequence of tasks are given in Table 1. Note that the changes that are made here are reflective of the retrospective analysis. One main area that was added to this sequence of tasks was the Cartesian Connection.

Table 1: Enacted and Revised Sequence of Tasks

Unit	*Enacted*	*Revised*
1	Multiple Representations of Equivalent Expressions	The "Cartesian Connection" in Graphs, Symbols, Tables, and Words Equivalent and Non-Equivalent Expressions in Graphs, Tables, Symbols, and Words
2	Equations are Equivalence Relations that are Sometimes, Always, or Never True	Equations are Equivalence Relations that are Sometimes, Always, or Never True
3	Solving Linear Equations with Multiple Representations	Identifying Solutions Sets of Linear Equations in Graphs, Tables, Symbols, and Words
4	n/a	Equivalent Equations have the Same Solution Sets

Ongoing analysis during the teaching experiment alluded to possible weaknesses in students' understanding of the Cartesian Connection (Fonger, 2012). As described by Moschovich et al (1993), this understanding is seen as an important pre-requisite skill for coming to understand the relationships between graphical, numeric, symbolic, and verbal representations of equations and solutions. The second area of the task structure that this affected is described in row 3 of Table 1; more attention needs to be given to the role of identifying solutions in multiple representations. Finally, the five-week teaching experiment was not long enough to fully test the mathematical topic of equivalent equations, thus is considered to be an additional unit to this sequence of activities (row 4 of Table 1).

Technique

Many of the techniques that were tested and revised in the experiment were process oriented with respect to the role of multiple representations (and supporting the development of representational fluency), and the role of using multiple tools. Table 2 gives a summary of these techniques; those with a double-asterisk (**) were added during the retrospective analysis of data.

Table 2: Techniques that Structured Classroom Activity with Revised Descriptions

	Techniques that Support Representational Fluency	
**	Translate	Create and interpret the meaning of a target representation with respect to a source representation of a different type.
	Transpos	Create and interpret multiple representations within one representation type.

e**	
Interpret	Convey the meaning of the act/result of creating a tool-based representation; a basic, quick remark, thoughtful but not deep.
Connect	Give a correct interpretation of an invariant feature across multiple representations or types.
Generalize**	Make a generalization across several representations or representation types (e.g., abstract notation $a(x + b) = ax + ab$).
Justify**	Representations are used to confirm or ascertain a particular result or conclusion; "use representations as justifications for other claims" (Sandoval, Bell, Coleman, Enyedy, & Suthers, 2000; reasoning must be present); formal/rigorous explanation, objective, based on set practices.
Techniques that Support the Coordination of Multiple Tools	
Anticipate	Predict the result of creating tool-based representations.
Act	Create a representation and possibly explain the process of how one works within or moves between tool-based representations or types.
Reflect	React to or think deeply about representations/representation types with respect to equivalence and/or equations; heavy thought, detailed in response, subjective and developmentally oriented.
Reconcile	Negotiate differences between CAS and paper-and-pencil representations.
CAS Check**	Use the CAS to check or verify paper-and-pencil representations (often times within Symbolic representation type).

The notion of an action-consequence principle (see the techniques at the bottom of Table 2) is proposed an appropriate way to coordinate the use of mathematics technology in which tool-based results are first predicted with paper-and-pencil, then executed with technology, then reflected upon (cf. Dick & Hollebrands, 2011). The notion of reconciling differences between CAS and paper-and-pencil representations is elaborated by Kieran & Saldanha (2008). The role of CAS to check paper-and-pencil results was added to the activity structure during the ongoing analysis when it became evident as a common classroom practice in which results were consistent between multiple tools (no reconciling was needed).

Theory

The sequence of instructional tasks was guided by a synthesis of several conceptual progressions into a single learning progression (see Table 3). Two studies were central: (a) research with secondary students who studied a unit on equivalence and equation solving with CAS and paper-and-pencil (Kieran & Drijvers, 2006) [elements A1, B-symbolic only, C, D1, in Table 3], and (b) research with middle grades students who studied a unit with a multi-representational lens on equivalence solving with graphing calculators and paper-and-pencil (Kieran & Sfard, 1999) [elements A2, B, D2 in Table 3]. The elements that were added to the learning progression during the teaching experiment to support student understanding of the learning goals are denoted with a single asterisk * in Table 3; all other components were determined a priori to the study.

Table 3: Hypothesized Modifications to a Learning Progression

Element	Description

A1	*Connecting and generalizing the quantitative, visual, and verbal with symbols*. Symbolic expressions generalize numeric, graphic, and verbal patterns by allowing for compact, abstract notation.
A2	*Different representations/representation types can signify the same object*. Different representations/representation types of the *same* linear expressions and/or equations signify the same relationship, pattern, or function from different yet complementary perspectives.
B	*Equivalence of expressions from multiple representations.* Expressions are equivalent if they define the same relationship, pattern, or function.
C	Domain and range restrictions may arise in contextual situations and should be considered when determining equivalence.
C1*	Role of Equal Sign: "=" assigns variables rules/names for patterns.
C2*	Role of Equal Sign: "=" expresses identity between equivalent expressions.
CC 1*	If a point *P* is on the line *L*, *P* makes the equation of *L* true.
CC 2*	If a point *P* makes the equation of *L* true, *P* is on the graph of *L*.
D1	*Solutions to equations can be determined by equality of expressions*. Linear equations are relations between linear expressions that are sometimes, always, or never equal in value. Thus linear equations have one, infinitely many, or zero solutions, respectively.
D2	*Solving equations in one variable is conceptualized as a comparison of two functions.* Linear equations in one variable such as $ax + b = cx + d$ for real valued parameters a, b, c, and d, can be solved for the variable x by comparing the functions $f(x) = ax + b$ and $g(x) = cx + d$ for the value of x that makes the equation $ax + b = cx + d$ true. Graphical, tabular, or symbolic methods can be used.
E	*Equivalence of equations*. Equations are equivalent if they have the same solution set. Represented graphically, solution sets of equivalent equations are x-coordinates of the intersection points in the coordinate plane. Represented in tables, solution sets of equivalent equations are the inputs for which the outputs are the same.

The aspects of the learning progression for which there is more to explore include the relationship in language between expressions, equations, and functions. During the teaching episodes the students became accustomed to using the language of an equation as being sometimes, always, or never true. Another aspect of the learning progression that was not tested with this particular group of students was that of equivalent equations [element E in Table 3]. This element is still included in the learning progression because of the importance of this topic as specified in CCSSI (2010).

Implications and Ongoing Research

Consistent with the goal of linking research and practice, the task-technique-theory framework was used as a tool to help support research-practice links in the design of instruction. The articulation of a research-based sequence of tasks can be used to guide instructional decisions about the implementation of mathematical content as espoused in CCSSI (2010). Moreover, for classroom environments that support the coordinated use of multiple tools, a focus on techniques can be used as a lesson design principle to guide instructional moves. This aspect of the emerging instructional theory is also consistent with the mathematical practice of using appropriate tools strategically. Techniques focused on the role of representations are also well suited for promoting a more conceptual understanding of mathematics as long as there is a strong emphasis on interpretation and connections.

The proposed instructional theory is characterized as emergent because based on the design research paradigm (Gravemeijer & Cobb, 2006), the testing of a learning progression should occur in several iterations in which the implementation of the first iteration elicits revisions and informs the next iteration of testing. The rationale for an iterative design is to build theory over time, not to just empirically tune “what works” but to elicit general design principles that can be used to inform other instructional design along a meaningful learning progression. Consistent with this research design, current research is underway that is focused on testing and refining the instructional theory. The task, technique, and theory elements are being integrated into a curriculum for pre-service secondary mathematics teachers. One way in which the ongoing research will extend the study discussed here is by the use of polynomial and rational expressions and equations for examining equivalence and equation solving, with more specific attention to the role of the equal sign with the newly emerging frameworks for understanding that knowledge (e.g., Matthews et al., 2012). Another way in which future research can extend and strengthen the current study is by investigating a broader range of mathematical standards so as to link topics from elementary grades on algebraic thinking to topics across the middle and high school that require justification and reasoning about equivalence and equation solving.

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