AN INVESTIGATION OF PREK–8 PRESERVICE TEACHERS' CONSTRUCTION OF FRACTION SCHEMES AND OPERATIONS

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Supporting students to build robust fraction schemes and operations is an enduring challenge in mathematics education. Recent research has explored a developmental trajectory of fractions schemes and operations constructed by upper-elementary and middle school students in an effort to support student learning. This study broadened the existing research by investigating PreK–8 preservice teachers' construction of fractions schemes and operations. This paper specifically explores data from PreK–8 preservice teachers regarding one scheme (Iterative Fraction Scheme) and way of operating (Three Level Units Coordination). Results focus on 13 special cases that disagreed with current conceptions of the developmental trajectory.

Keywords: Learning Trajectories (or Progressions); Mathematical Knowledge for Teaching; Rational Numbers; Teacher Education-Preservice

Introduction and Objectives

In light of the enduring challenge that understanding fractions concepts places on PreK–8 preservice teachers (PSTs), our research project sought to validate a developmental trajectory of fractions schemes and operations specifically for PSTs. In particular, this study extends current research that has worked to establish this trajectory for elementary and middle school students (Hackenberg, 2007; Norton & Wilkins, 2012; Steffe & Olive, 2010). For this study PSTs responded to tasks designed to determine the fractions schemes and operations with which each participating PST seemed to operate.

The initial goal was to validate the established trajectory with a new population. Upon analysis of the data, the trajectory seemed consistent with previous research; that is, each lower level of fractional understanding appeared to be a prerequisite to higher levels of understanding. However, upon further analysis, 13 special cases emerged that deviated from this trend. In these 13 cases, PSTs demonstrated ways of operating that aligned with an Iterative Fraction Scheme (IFS) while simultaneously lacking the prerequisite operation of Three Level Units Coordination (3UC). This paper focuses on our work examining these special cases. We worked to answer the research question "Must PSTs interiorize 3UC before constructing an IFS?" Although studies have shown that, without 3UC, elementary and middle school-aged students cannot develop an IFS (e.g. Hackenberg, 2007; 2010), it appeared possible for PSTs to do so.

Background and Theoretical Framework

PSTs and Fractions

Many studies document PSTs' difficulties with making sense of fraction concepts and fraction computation, particularly related to fraction division (e.g., Ball, 1990; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Newton, 2008; Tirosh, 2000; Van Steenbrugge, Lesage, Valcke, &

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Desoete, 2014). One possible cause for this enduring challenge is the prevalence of part-whole thinking in U.S. mathematics curriculum (Newton, 2008; Watanabe, 2007; Yang, Reys, & Wu, 2010). Part-whole thinking is a common way to define fractions: The fraction *m*/*n* is defined as *m* equal-sized parts out of *n*, where a total of *n* parts make up the whole (where *m* and *n* are positive integers). Using this fraction scheme, 3/4 would be thought of as 3 equal-sized parts out of 4.

Previous studies have established that PSTs rely primarily on part-whole thinking (Newton, 2008; Sowder, Bedzuk, & Sowder, 1993; Tirosh et al., 1998; Zhou, Peverly, & Xin, 2006). Although part-whole reasoning can provide an initial understanding of fractions, research has documented the limitations of part-whole reasoning (Mack 2001; Olive & Vomvoridi, 2006; Saenz-Ludlow 1994). One of the major limitations of part-whole reasoning relates to understanding improper fractions. Students typically struggle to reason with fractions greater than 1 when the only way they know to think about fractions is part-of-a-whole (Thompson & Saldanha, 2003). When 3/5 only makes sense as three parts out of five, it is difficult to make sense of 7/5, or seven parts out of five.

Besides the reliance on part-whole thinking, another possible cause for PSTs' issues with thinking about improper fractions as numbers is difficulty with coordinating multiple levels of units (Hackenberg, 2007). Construction of improper fractions as numbers requires the *interiorization* of coordinating of three levels of units (Hackenberg, 2007, 2010; Norton & Wilkins, 2012). *Interiorization* is the reorganization of internalized actions as an assimilatory structure; students who have interiorized actions (as operations) do not need to carry them out in activity (i.e., they can be taken as given prior to activity) (Olive, 2001).

Table 1: Fraction Schemes

Schemes	Associated Mental Actions
Part-Whole Fraction Scheme (PWS)	Producing *m*/*n* by partitioning a whole into *n* pieces and disembedding *m* of those pieces.
Partitive Unit Fraction Scheme (PUFS)	Determining the size of a unit fraction relative to a given unpartitioned whole by *iterating* the unit fraction to produce a continuous partitioned whole.
Partitive Fraction Scheme (PFS)	Determining the size of a proper fraction relative to a given unpartitioned whole by *partitioning* the proper fraction to produce a unit fraction and *iterating* the unit fraction to reproduce the proper fraction and the whole.
Reversible Partitive Fraction (RPFS)	Reproducing the whole from a proper fraction of it by *partitioning* the fraction to produce a unit fraction and *iterating* that unit fraction the appropriate number of times. Note that the action of partitioning implicitly involves splitting because partitioning is used to reverse the iterations of a unit fraction (e.g., 3/5 as three iterations of 1/5).
Iterative Fraction Scheme (IFS)	Reproducing the whole from an improper fraction of it by partitioning the fraction to produce a unit fraction and iterating that fraction unit fraction the appropriate number of times. Note that, in addition to splitting, this way of operating implicitly involves coordinating three levels of units: the unit fraction, the improper fraction, and the proper fraction contained within it.

Fractions Schemes and Operations

Steffe and Olive (2010) suggest a learning trajectory for students' development of fractions knowledge in terms of schemes and operations. The hierarchy of schemes is found in Table 1. In addition, two ways of operating also essential to this framework: 1) Splitting, which is the mental action of simultaneous *partitioning* and *iterating*, and 2) mental actions associated with Three Level Units Coordination (3UC).

Methods

Context and Participants

The participants were 109 undergraduates enrolled in the first of three required mathematics courses for PreK–8 PSTs. The focus of this first course was number concepts and operations, including fractions. The 109 students include only those taking the course for the first time. Of the 109 participants, about one-half were freshman (52%), about one-third were sophomores (31%), and the rest were upperclassmen (17%).

Data Collection

The participants were given a fractions schemes and operations assessment at the beginning of the semester, specifically before any instruction related to fractions or fraction computations. The assessment contained four items associated with each of the seven fractions schemes and operations (see Table 1), resulting in a total of 28 items. Each item was designed to provoke a response that would indicate whether or not the student had constructed that particular scheme or operation. These items were developed for and previously used with upper elementary and middle school students (Norton & Wilkins, 2012; Wilkins & Norton, 2011). However, PSTs have and tend to use knowledge that upper elementary and middle school students do not necessarily automatically use (e.g., fraction division algorithms). Therefore, to obtain a better understanding of the PSTs' ways of operating, the participants were also asked to provide a brief written explanation for their responses.

Two coders independently rated the responses using both the written work on the assessment and the brief explanations provided, according to Norton and Wilkins (2009, p. 156). Each item was given a score of 0, 0.4, 0.6, or 1 based on the amount of evidence observed for a given scheme or operation (see Table 2).

Table 2: Scoring Rubric

Score	Evidence Shown
0	Strong counterindication that the PST could operate in a manner compatible with that particular scheme or operation
0.4	Weak counterindication that the PST could operate in a manner compatible with that particular scheme or operation
0.6	Weak indication that the PST could operate in a manner compatible with that particular scheme or operation
1	Strong indication that the PST could operate in a manner compatible with that particular scheme or operation

Each coder summed the four individual item scores, resulting in an overall raw score between 0 and 4for each scheme or operation. The overall raw scores were then used to infer whether or not the PST had constructed that particular scheme or operation. Overall raw scores greater than or equal to 3 indicated that the PST had constructed that particular scheme or operation. Overall raw scores less than or equal to 2 indicated that the PST had not constructed that particular scheme or operation.

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Overall raw scores between 2 and 3 required each coder to infer from all the given information whether or not the PST had constructed that particular scheme or operation. If a disagreement occurred between the two raters, the raters reexamined all the relevant information together to reach a consensus. (The average kappa statistic for measuring inter-rater reliability for 3UC was .81 and for IFS was .93.)

Data Analysis

First, descriptive statistics were calculated in order to compare the percentages of PSTs that had constructed each of the different schemes and operations. For this paper, we focus in particular on Three Level Units Coordination (3UC) and the Iterative Fraction Scheme (IFS).

Second, data were entered into 2 × 2 contingency table to examine the hypothesized association between 3UC and IFS. The gamma statistic, G, was used to test the magnitude of the association (Siegel & Castellan, 1988). We were specifically interested in testing whether the interiorization of 3UC preceded the construction of an IFS. Based on prior research (Hackenberg, 2007; 2010), we predicted a direct or positive association between the interiorization of 3UC and the construction of an IFS, specifically that the interiorization of 3UC occurs prior to the construction of an IFS. If the data associated with 3UC and IFS are consistent with this hypothesis, then we would find a positive direct association (i.e., $G> 0$) and a *weak monotonic relationship*. A weak monotonic relationship is characterized by a staircase pattern in the contingency table, with data falling predominantly in the diagonal and lower left cell. Because the G statistic is a symmetrical measure of association it does not by itself provide evidence of developmental order among the schemes and operations.

Following procedures outlined by Wilkins and Norton (2011) we tested for developmental order among the schemes and operations by first visually examining the table for evidence of a staircase pattern. Empirically, using a binomial test, we tested whether the difference in the number of cases in the off-diagonal cells was in the hypothesized direction and different from what would be expected by chance. We hypothesized a direct (positive) association between 3UC and IFS and also hypothesized a developmental order. For this hypothesis we used a one-tailed gamma and binomial test.

Results

In Table 3, descriptive statistics for 3UC and IFS are presented. Less than half of the PSTs had interiorized the coordination of three levels of units (47%). About a quarter of the PSTs had constructed an IFS (27%).

Table 3: Descriptive Statistics

Scheme/Operation	Percentage	*SD*
3UC	47%	0.50
IFS	27%	0.44

Based on our hypothesis, PreK–8 PSTs should interiorize 3UC prior to constructing an IFS. Table 4 presents the frequencies of PSTs' construction of an IFS and coordination of three levels of units. Overall, no association between the coordination of three levels of units and the construction of an IFS was found ($G = .23$, $p =.15$, one-tailed). An examination of the off-diagonal cases ($n = 48$) found 13 PSTs who had constructed an IFS prior to interiorizing the coordination of three levels of units. This is a relatively large number of cases that counter the hypothesis. However, the distribution of the off-diagonal cases was statistically beyond chance; exact binomial, $p = .001$ (one-tailed).

Table 4: Frequency of 3UC and IFS Scores

	IFS		
3UC	0	1	Total
0	45	13	58
1	35	16	51
Total	80	29	109

Note. $G = .23$, $p = .15$, one-tailed; Exact Binomial $p = .001$ (one-tailed).

Because the result from the gamma statistic is inconsistent with the theory relating 3UC and an IFS (Hackenberg, 2007; 2010), it is important to further examine the 13 students found to have constructed an iterative fraction scheme prior to interiorizing the coordination of three levels of units. One point of concern is that the fractions tasks used in the assessment were designed for upper elementary and middle school students. As previously discussed, PSTs have and use knowledge that upper elementary and middle school students do not automatically employ. For example, by using algorithms for dividing fractions or finding equivalent fractions, PSTs' procedural responses to the 3UC and IFS tasks, as well as their written explanations, may actually mask evidence providing an indication for (or evidence providing a counterindication against) interiorizing 3UC or constructing an IFS. For both the 3UC and IFS tasks, the 13 PSTs responses were re-examined to find patterns in their thinking and representations.

One observation found was that many of the 13 PSTs used either fraction division or equivalent fractions to answer the 3UC tasks. For example, Figure 1 shows how one PST used fraction division to answer a 3UC task. In her explanation, the PST described her thinking: "The pizza shows 3/4 of a pizza and each person wants 1/8 so I divided 3/4 by 1/8. I found the reciprocal making it 3/4 ÷ [sic] 8/1 and found that 6 people could get 1/8 of the pizza." Although her answer is correct, her procedural work and written description do not provide clear evidence that this PST is actually coordinating three levels of units.

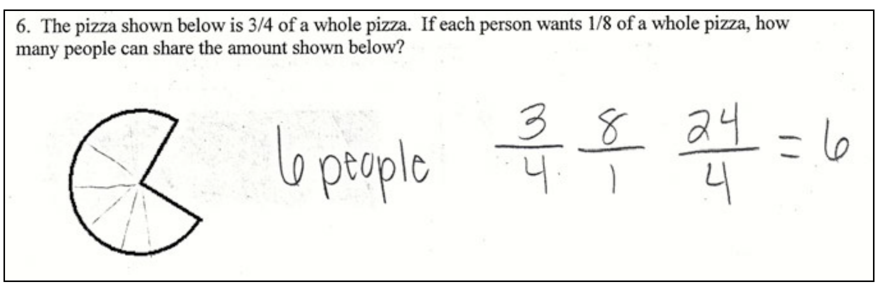

Figure 1. PST used fraction division to answer a 3UC task.

Figure 2 exhibits how another PST solved a 3UC task, this time using equivalent fractions. In her written work, she explained that she wanted to find out "how many eighths are in 3/4 so I multiplied top and bottom by 2 to reach eighths and I got 6/8. Therefore 6 people can have 1/8 of the 3/4 pizza." Again, the procedural work and written explanation do not provide evidence for or against this PST having interiorized 3UC.

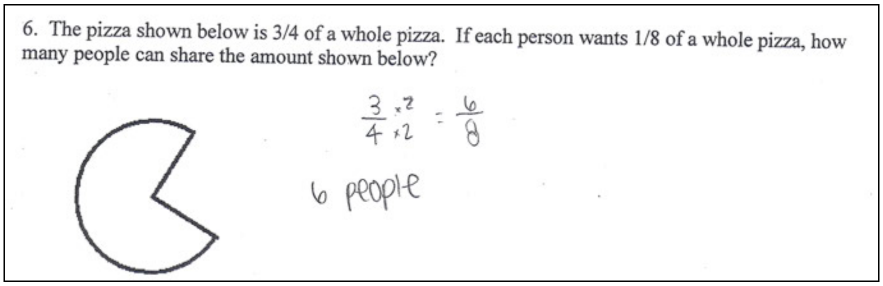

Figure 2. PST used equivalent fractions to answer a 3UC task.

Another observation found in the responses to IFS items was that over half of the 13 PSTs changed the given improper fraction to a mixed number to find an answer. Although these PSTs were able to determine the correct answer, it seemed that they would have been unable to do so without converting to a mixed number based on their written explanations. As an example, one PST (see Figure 3) wrote in her explanation, "I changed 7/3 to a mixed fraction and saw it was roughly twice the size as the candy bar so I split it into thirds and counted 3/3 to make one." If this PST had constructed an IFS, she would likely respond by partitioning the given stick into seven equal pieces and taking three of those pieces to represent the whole candy bar—a more efficient way of operating. In this illustrative work, it seems as though the PST relied on the mixed number of 2 1/3 to solve the task instead of considering the improper fraction of 7/3 as a number in its own right.

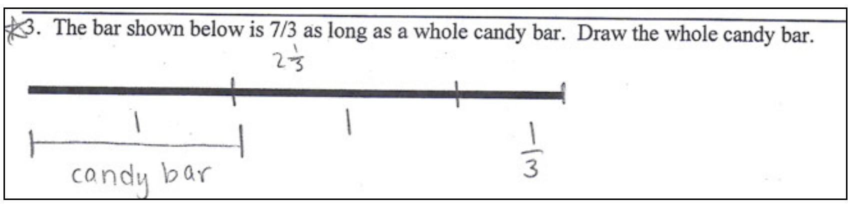

Figure 3. PST used a mixed number to answer an IFS task.

Also, almost half of the 13 PSTs examined demonstrated some level of confusion in their responses to IFS items. Even when they provided correct representations and answers, these PSTs exhibited a lack of confidence in their work. For example, one PST explained, "I'm having trouble understanding the amounts when the given amount is over 1." Another PST expressed confusion when the given diagram (which represented an improper fraction) was a half circle: "I got confused looking at the diagram because the picture doesn't look bigger than a whole." The confusion unveiled in responses like these raise further questions about these PSTs' construction of an IFS.

The analysis of the 13 PSTs' responses suggests that some PSTs may have actually interiorized 3UC, but their use of procedures and algorithms potentially mask the coordination of the three levels of units. In addition, some issues related to the IFS tasks, such as relying on mixed numbers, may have been overlooked and resulted in a false identification of that PST constructing IFS. As such, our results call for further investigation.

Discussion

We found that assessing the interiorization of Three Level Units Coordination (3UC) in PreK–8 preservice teachers is challenging. Primarily this challenge arose from the PSTs' automatized mathematical procedures that upper elementary and middle school students may still be in the process of learning. Instead of having to coordinate three levels of units on these tasks, PSTs may just be using their procedural knowledge for dividing fractions or finding equivalent fractions. These computational procedures mask evidence for or against the interiorization of 3UC. While PSTs may

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be able to use a procedure (see Figures 1 and 2) to find a correct answer, they often do not demonstrate evidence for or against their ability to view 3/4 of a whole pizza as three 1/4 pieces, where four such pieces would make up a whole pizza, and that each 1/4 piece contains two 1/8 pieces.

Likewise, we found assessing whether or not PreK–8 preservice teachers have constructed an iterative fraction scheme (IFS) to also be challenging. Two reasons for this challenge are the PSTs' use of mixed numbers and lack of confidence. Hackenberg (2007) claims that when students cannot consider an improper fraction as a number in its own right, and rather must change the number to a mixed number, then the student has not constructed an IFS. Some of the 13 students found to have constructed an IFS prior to interiorizing 3UC may have "fooled" us into believing they had constructed an IFS even though they relied on mixed numbers to reach (and represent) their solution. In addition, the PSTs were often confused about the problem statement as well as their solutions to IFS items. In retrospect, we wonder whether we should have inferred that a PST has constructed an IFS when they exhibit confusion and lack confidence in their solutions.

From part of the analysis of our data, it seems that some PreK–8 preservice teachers may have constructed IFS without the interiorization of 3UC. However, because of the problematic assessment of these items, as noted above, it is difficult to make a definitive conclusion. Because PSTs' difficulties with fraction concepts and fraction computations is an enduring challenge for researchers and teacher educators alike, this current research should be expanded. One starting point is to redesign the items used in our current assessment (which were originally designed for upper elementary and middle school students) so that PSTs cannot use procedures and algorithms to solve; instead, they must rely on their constructed schemes and interiorized operations. Another expansion of our research is to conduct clinical interviews with the PreK-8 preservice teachers to better understand whether or not they had truly interiorized 3UC or constructed an IFS.

In addition to the advancement of research, changes should also be considered in required mathematics courses for PreK–8 preservice teachers. For example, instead of using language that reinforces part-whole thinking (i.e., describing the fraction 4/5 as four equal-sized parts out of five), instructors can emphasize language that encourages a more iterative way of thinking (i.e., describing the fraction 4/5 as 4 equal-sized parts, each of which is 1/5 of the whole). In addition, instructors can incorporate more instructional tasks and activities that involve improper fractions, such as asking PSTs to model and describe improper fractions using representations or manipulative materials such as Pattern Blocks. These practices may help PSTs move towards interiorizing the coordination of three levels of units and constructing an IFS.

With the call for students as young as fourth and fifth grade to operate with higher-level schemes and operations (CCSSO, 2010), it is imperative that future PreK–8 teachers also be able to operate with higher-level schemes and operations. Both research and practice can help to accomplish this goal.

References

Ball, D. L. (1990).The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*, 449–466.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers give up too easily? *Journal for Research in Mathematics Education*, *23*(3), 194-222.

CCSSO (Council of Chief State School Officers). (2010). *Common core state standards.* Retrieved from http://corestandards.org

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, *26*(1), 27-47.

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction*, *28*(4), 383-432.
Mack, N.K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education*, *32*(3), 267–296.
Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal*, *45*(4), 1080-1110.
Norton, A., & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *Journal of Mathematical Behavior, 28*(2/3), 150–161.
Norton, A., & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education*, *43*(5), 557-583.
Olive, J. (2001). Children's Number Sequences: An Explanation of Steffe's Constructs and an Extrapolation to Rational Numbers of Arithmetic. *The Mathematics Educator, 11*(1), 4-9.
Olive, J., & Vomvoridi, E. (2006). Making sense of instruction on fractions when a student lacks necessary fractional schemes: The case of Tim. *Journal of Mathematical Behavior*, *25*(1), 18-45.
Sáenz-Ludlow, A. (1994). Michael's fraction schemes. *Journal for Research in Mathematics Education*, *25*(1), 50-85.
Siegel, S., & Castellan, N. J. (1988). *Nonparametric statistics for the behavioral sciences*. New York, NY: McGraw-Hill Book Company.
Sowder, J. T., Bedzuk, N., & Sowder, L. K. (1993). Using principles from cognitive psychology to guide rational number instruction for prospective teachers. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 239–259). Hillsdale: Lawrence Erlbaum.
Steffe, L. P., & Olive, J. (Eds.) (2010). *Children's fractional knowledge*. New York: Springer.
Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics*, (95–113). Reston, VA: National Council of Teachers of Mathematics.
Tirosh, D., Fischbein, E., Graeber, A. O., & Wilson, J. W. (1998). *Prospective elementary teachers' conceptions of rational number*. Retrieved from http://jwilson.coe.uga.edu/Texts.Folder/Tirosh/Pros.El.Tchrs.html
Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31*(1), 5-25.
Van Steenbrugge, H., Lesage, E, Valcke, M., & Desoete, A. (2014). Preservice elementary school teachers' knowledge of fractions: A mirror of students' knowledge. *Journal of Curriculum Studies*, *46*(1), 138-161.
Watanabe, T. (2007). Initial treatment of fractions in Japanese textbooks. *Focus on Learning Problems in Mathematics, 29*(2), 41–60.
Wilkins, J. L. M., & Norton, A. (2011). The splitting loope. *Journal for Research in Mathematics Education*, *42*(4), 386-416.
Yang, D. C., Reys, R. E., & Wu, L. L. (2010). Comparing the development of fractions in the fifth- and sixth-graders' textbooks of Singapore, Taiwan, and the USA. *School Science and Mathematics*, *110*(3), 118-127.
Zhou, Z., Peverly, S. T., & Xin, T. (2006). Knowing and teaching fractions: A cross-cultural study of American and Chinese mathematics teachers. *Contemporary Educational Psychology*, *31*, 438–457.

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.