REASONING QUANTITATIVELY TO DEVELOP INVERSE FUNCTION MEANINGS

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Researchers have argued that students can develop foundational understandings for a variety of mathematical concepts through quantitative reasoning. I extend this research by exploring how students' quantitative reasoning can support them in developing meanings for inverse relations that influence their inverse function meanings. After summarizing the literature on students' inverse function meanings, I provide my theoretical perspective, including a description of a quantitative approach in the context of inverse relations. I then present one student's activity in a teaching experiment designed to support her in reasoning about a relation and its inverse as representing the same relationship. The student's quantitative reasoning supported her in developing productive meanings for inverse function, although this required her to reorganize her understanding of various mathematical ideas.

Keywords: Cognition; Teacher Education-Preservice; Design experiments

Researchers have indicated students can leverage reasoning quantitatively to develop meanings for various mathematical topics before developing more formal mathematical understandings (Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Johnson, 2012). Researchers using a quantitative reasoning lens have also provided important insights into students' learning of a variety of secondary mathematics topics including specific function classes (e.g. linear functions (Johnson, 2012; P. W. Thompson, 1994) and exponential functions (Ellis et al., 2012)). A natural extension of this body of research is to explore how students' quantitative reasoning influences their notions of relations (or functions) and inverse relations prior to and concurrently with thinking about specific function classes. In this report, I summarize the research on students' inverse function meanings then propose ways of thinking that have the potential to support students in developing productive inverse relation and inverse function meanings. I present important aspects of a student's activities from a semester-long teaching experiment designed to support her in developing such meanings. I conclude with implications stemming from this work and directions for future research.

Research on Inverse Function

Vidakovic (1996) presented a genetic decomposition for inverse function (i.e. a description of how students might learn a concept, including methods for constructing their schemes). She proposed that students develop inverse function schemas in the following order: function, composition of functions, then inverse function. She conjectured students could coordinate all three schemas and develop inverse function meanings through this coordination. Whether implicitly or explicitly, many researchers (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Vidakovic, 1997) who have examined students' inverse function meanings have maintained an emphasis on composition of functions as critical to students developing productive inverse function meanings. However, these same researchers noted that students often carry out techniques for successfully determining representations of inverse functions (e.g., determining an inverse function analytically) without connecting these techniques to function composition. As a consequence, students (and teachers) hold compartmentalized inverse function meanings, typically related to executing specific actions in analytic or graphing situations (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Paoletti, Stevens, Hobson, LaForest, & Moore, in

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press). Moreover, Paoletti et al. (in press) reported that pre-service teachers, when given a function meant to represent a context, struggled to interpret the contextual meaning of the inverse function they constructed. Collectively, these researchers' findings suggest that current approaches to inverse function have been ineffective in supporting students in developing productive inverse function meanings. Complicating the matter, and as I argue in more detail below, researchers predominately treat students construction of 'inverse function' meanings as distinct from their function understandings (e.g., Vidakovic's genetic decomposition), as opposed to approaching 'inverse function' as developing hand-in-hand with 'function'.

Theoretical Framing

I explore the possibility of supporting students developing inverse meanings via reasoning about quantities and relationships between quantities (i.e. reasoning quantitatively). A *quantity* is a conceptual entity an individual constructs as an attribute of an object or phenomena that allows a measurement process (P. W. Thompson, 1994). As an individual associates two varying (or non-varying) quantities, she can construct *quantitative relationships* (Johnson, 2012; P.W. Thompson, 1994); an individual engages in *quantitative reasoning* as she constructs and analyzes these relationships (P. W. Thompson, 1994).

Using a quantitative reasoning lens, I conjectured that if a student constructed a relationship between two quantities (e.g., quantities A and B) that did not entail some conceived causation between the quantities, the student could choose one quantity to be the input of a relation (e.g., A input, B output) while anticipating that the inverse relation would involve choosing the other quantity as the input (e.g., B input, A output). With respect to graphing relations and inverse relations in the Cartesian coordinate system, a student who understands relations in ways compatible with this could interpret a single graph as representing both a relation and its inverse. Engaging in such reasoning requires the student to anticipate choosing either axis as representative of an input quantity; researchers (Moore & Paoletti, 2015; A. G. Thompson & Thompson, 1996) have defined such reasoning as bidirectional reasoning. Although this type of reasoning may seem trivial, Moore, Silverman, Paoletti, and LaForest (2014) illustrated that students are often restricted to reasoning about the quantity represented along the horizontal axis as the input.

By focusing on a relationship between quantities, the 'function-ness' of a relation and its inverse is not critical. A student can describe a relation and its inverse without (necessarily) being concerned if either represents a function. Moreover, the student understands that the choice of input-output quantities does not influence the underlying relationship that the associated functions or relations describe. Whereas this approach does not foreground function (or composition of functions) as critical to developing inverse meanings, I conjecture a student who develops understandings compatible with those described would have little difficulty making sense of the formal definition of inverse function that relies on composition of function (e.g., understanding if $B = f(A)$ and $A = f^{1}(B)$, then $f(f^{1}(B)) = f(A) = B$ and $f^{-1}(f(A)) = f^{1}(B) = A$).

Methods

I conducted a semester-long teaching experiment with two undergraduate students, Arya and Katlyn (pseudonyms). I focus this report on Arya's activity. Arya was a junior who had successfully completed a calculus sequence and at least two additional courses beyond calculus. The teaching experiment consisted of three individual semi-structured task-based clinical interviews (per student) (Clement, 2000) and 15 paired teaching episodes (Steffe & Thompson, 2000). I used clinical interviews to explore Arya's function inverse meanings without intending to create shifts in her meanings. I used the teaching episodes to pose tasks and questions that I conjectured might perturb Arya's meanings, leading her to make accommodations to her meanings to resolve her perturbations.

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I used the combination of clinical interviews and teaching episodes to explore Arya's mathematical activity, to build models of her mathematics, and to investigate the mathematical progress Aryamade over the semester (Steffe & Thompson, 2000).

In order to analyze the data, I used open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998) in combination with conceptual analysis(P. W. Thompson, 2008)to develop and refine models of Arya's mathematics. Initially, I analyzed the videos identifying episodes of Arya's activity that provided insights into her meanings. Using these identified instances, I generated tentative models of her mathematics that I tested by searching for activity that corroborated or refuted my models. When Arya exhibited novel activity that contradicted my models, hypotheses were made to explain this activity including the possibility that this new activity indicated fundamental shifts in her operating meanings. Through this iterative process of creating, refining, and adjusting hypotheses of Arya's meanings, I was able to not only characterize her thinking at a specific time or situation, but I was also able to explain transitions in Arya's meanings throughout the teaching experiment.

Task Design

I focus this report primarily on one task (*Graphing sine/arcsine task*, Figure 1), which a research team designed to support the pair of students in developing productive inverse relation meanings via reasoning bidirectionally. Relevant to this report, the first two parts of this task involve the students creating graphs of the sine (Graph 1) and arcsine function (Graph 2). The third prompt asks the students to consider how they could use Graph 1 to represent the arcsine function. The prompt also asks the students to consider if Graph 1 and Graph 2 represent "the same relationship." I conjectured by asking the students to foreground the "relationship" represented by both graphs, they might engage in reasoning bidirectionally in order to conceive Graph 1 and Graph 2 as representing both the sine and arcsine functions (or relations).

Graph 1:	Create a graph of the sine function with a domain of all real numbers. What is the range?
Graph 2:	Using **covariation talk**, create and justify a graph of the arcsine (or inverse sine) function.
Prompt 3:	Can you alter (**do not draw a new graph**) Graph 1 such that it represents the graph of the arcsine function? Does this graph convey the same relationship as the second graph? How so or how not?

Figure 1: Graphing sine/arcsine task

Results

For brevity's sake, I highlight important instances in Arya's activity in order to describe her thinking including shifts in her thinking. I first describe Arya's activity (most relevant to this report) during the initial clinical interview in order to characterize her meanings prior to the teaching episodes. I then provide data from four consecutive teaching episodes in which Arya addressed the prompts in the *Graphing sine/arcsine task* for various relations.

Results from the initial clinical interview

When given a function's graph and asked to determine a graph of the inverse function, Arya switched the coordinate values (e.g., a point (a, b) from the original curve became (b, a) on the inverse curve). When asked to determine the inverse of a function represented analytically, Arya switched the variables and solved for the previously isolated variable (heretofore referred to as switched-and-solved) (e.g., the inverse of $y = x + 1$ was $y = x – 1$). I also gave Arya a function defined analytically that converted degrees Fahrenheit to degrees Celsius (i.e., $C(F) = (5/9)(F – 32)$ and asked her to represent the inverse function. She switched-and-solved obtaining $C^{-1}(F) = (9/5)F + 32$. When asked to interpret the meaning of the inverse equation, Arya considered again switching the variables (e.g., $F(C) = (9/5)C + 32$), but rejected the resulting equation because it defined the same relationship between degrees Fahrenheit and Celsius as the original equation and function. I

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inferred from Arya's activity that she anticipated a function and its inverse function represented relationships that differed in some way other than a choice of defining input-output quantities (e.g., having different graphs or defining a different relationship).

Also of note from the interview, Arya exhibited activity in multiple problems that I took to indicate she was restricted to reasoning about the horizontal axis as representing a function's input. For instance, when given the graphs in Figure 2 and asked "Are these graphs the same or different?" Arya argued, "[the graphs are] showing the same thing but in a different way… this [*Figure 2a*] is what is happening to distance as time is going on." Then describing Figure 2b, Arya stated, "As you change your distance… the time is moving forward." In this and other cases, Arya maintained considering the horizontal axis as representing a function's input or the independent quantity of a relationship.

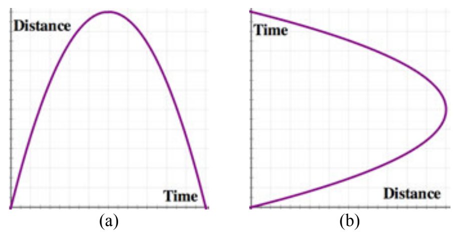

Figure 2: Double parabolas problem: Are the graphs the same or different?

Considering sine and arcsine

Nine days prior to the first teaching episode exploring the *Graphing sine/arcsine task*, Arya and Katlyn constructed the sine function as a relationship between an angle measure (input) and a vertical segment length above the horizontal diameter measured relative to a circle's radius (output) in a circular motion context (see Moore (2014)). Upon my giving the students the *Graphing sine/arcsine task*, they reproduced the graph of the sine function (see Figure 3a, Graph 1). Arya then leveraged her understanding of switching the coordinate values to graph the arcsine relation (e.g., the point ($\pi/2$, 1) became (1, $\pi/2$), see Figure 3a, Graph 2). The students labeled the horizontal and vertical axes in Graph 2 'vertical distance' and 'angle measure', respectively. I then questioned the students about "what [the] two graphs are representing?" Arya responded, "They're showing the same relationship, but this [*pointing to Graph 2, Figure 3a*] shows… if you're changing your vertical distance on your graph, what [*pointing to θ-label on the vertical axis*]radian measure that corresponds to. And this shows [*pointing to Graph 1, Figure 3a*] if you're changing your angle measure what vertical distance that corresponds to." As during the *Double parabolas problem*, Arya *described* the graphs as representing the same "relationship," but her interpretation of each graph relied on the horizontal axis as representing the input quantity (e.g., the quantity that she first envisioned varying or caused the other quantity to vary).

With both students content in their explanations of the two graphs, I asked them to consider the third prompt with the hopes of raising the underlying difference between their understandings of the graphs. Katlyn first wrote the analytic equation $\sin^{-1}(y) = \theta$ near Graph 1 and described interpreting Graph 1 with the vertical and horizontal axis representing the input and output quantity, respectively, of the arcsine relation. Arya responded, "I don't know if that, can you do that?" Katlyn's claim contradicted Arya restricting a function's input to the horizontal axis. As the interaction continued, Arya attempted to refute Katlyn's reasoning. But, as she attempted to do so, Arya continually returned to her understanding that Graph 1 represents the same distance-angle measure pairs regardless of which axis is denoted as a function's input. She concluded, "I don't see anything mathematically incorrect. I don't see it."

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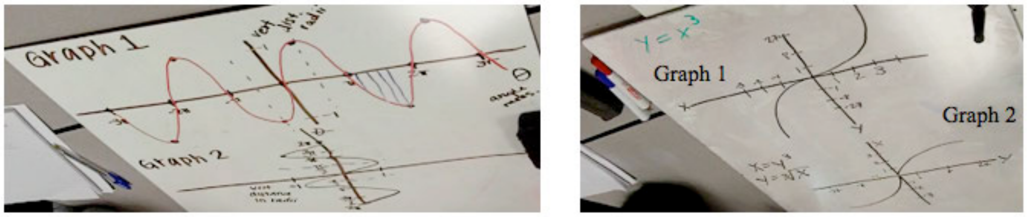

(a) (b)

Figure 3: Graphs of (a) the sine and arcsine functions and (b) a cubicand its inverse

By focusing on both graphs as representing, "Vertical distance and angle measure… the relationship between the two," Arya reorganized her meaning for interpreting graphs. Specifically, as Arya addressed the third prompt in the *Graphing sine/arcsine task* and Katlyn's claim, she had to consider whether graphs unquestionably represented the input quantity on the horizontal axis or if this was a common practice of graphing. Once Arya understood that graphs could be interpreted with either axes as representing the input (i.e., reasoned bidirectionally with respect to axes), she understood a single graph as representing both a relation and its inverse. By the end of the second teaching episode, Arya exhibited this understanding multiple times with respect to both Graph 1 and Graph 2, leading me to conjecture she had constructed the sine and arcsine relations as representing the same relationship between angle measure and vertical distance.[1] Moreover, she understood that a graph of the sine relationship simultaneously represented a graph of the arcsine relationship, and vice versa.

Considering a decontextualized function

I designed the third teaching episode to explore how Arya might extend her reasoning with the sine and arcsine relations to a relation or function represented by a decontextualized equation ($y(x)= x^3$). For instance, I was unsure if she would continue to reason about a relation and its inverse as representing the same underlying relationship, particularly when graphed, or if she would encounter perturbations due to the different context and the chance that she might use her switch-and-solve technique (on the previous task they maintained a quantitative referent for each variable rather than switching the variables). After graphing $y= x^3$ (Figure 3b, Graph 1), the pair switched-and-solved to obtain the inverse rule $y=x^{1/3}$. They were unsure how to graph this equation so I suggested they recall their activity from the previous sessions. In response, they labeled the horizontal axis *y,* the vertical axis *x*, and they constructed Graph 2 (Figure 3b) by considering how *y* changed (along the horizontal axis) for changes in *x* (along the vertical axis) so that they maintained the same *x*-*y* relationship of Graph 1. That is, when drawing Graph 2, their focus was not on the equation $y=x^{1/3}$, but instead the relationship between the varying values *x* and *y* as depicted in Graph 1.Arya argued, "all the same information is in both graphs." Compatible with the outcome of the prior two teaching episodes, Arya's (and Katlyn's) graphing activity indicated that she anticipated that a relation (or function) and its inverse represented the same relationship between quantities that could be represented graphically in multiple ways.

Due to my perceived discrepancy in their Graph 2 and the equation they had determined, I asked Arya to write an equation for Graph 2. She pointed to the equation $y = x^{1/3}$ but quickly noticed Graph 2 (Figure 3b), as labeled, did not represent an equivalent relationship between the varying values *x* and *y*. Because of this, Arya relabeled the vertical axis *y* and the horizontal axis *x* so that Graph 2 represented, "This equation [*pointing to* $y = x^{1/3}$]." Although Arya's newly labeled graph represented the equation $y=x^{1/3}$, she immediately experienced another perturbation. Given her new axes labels, she noted that Graph 1 and Graph 2 did not represent the same relationship between the varying values *x* and *y*. Hence, Arya realized that her switching-and-solving activity was inconsistent with

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her activity in the previous teaching episodes where she maintained the relationship between the quantities (i.e., variables).

As Arya was unable to reconcile her perceived inconsistency between switching-and-solving and maintaining the relationship between quantities (or variables) in both Graph 1 and Graph 2, I directed her to address the third prompt believing this may support her in considering the relation in *Graph 1* as simultaneously representing the inverse relation, and that she might then note that her Graph 2 was merely the result of using variables arbitrarily. Consistent with her activity interpreting Graph 1 in Figure 3a as the arcsine relationship, Arya described two ways of interpreting Graph 1 in Figure 3b by considering either the horizontal or vertical axis as her input. Although Arya had no difficulty reasoning bidirectionally with respect to the axes in Graph 1, this did not support her resolving the differences she perceived between Graph 1 and Graph 2 (Figure 3b) due to the discrepancy introduced by her use of variables.

Contextualizing the function

As Arya began to question the validity of her activity when graphing the arcsine relation, and based on my interpretation that she did not realize that switching-and-solving requires using variables arbitrarily, I attempted to give contextualized meanings to the variables in order to support Arya in reflecting on her use of variables. I rewrote the given equation as $V = s^3$ and asked Arya to consider the equation as representing the volume of a cube (V) for a given side length (s) with the caveat that we could have negative side length and volume values. When considering the context, Arya labeled both Graph 1 and Graph 2 in a way that maintained both the relationship between volume and side length and the variable referents; in Graph 1 she labeled the horizontal axis *side length* and the vertical axis *volume* and in Graph 2 she labeled the horizontal axis *volume* and the vertical axis *side length*. Although Arya maintained the relationship between side length and volume in both graphs, this did not alleviate her perturbation as she remained unsure how this related to her switching-and-solving activity.

To support Arya in considering a way to relate her activity maintaining the relationship between quantities (and maintaining variable referents) with her switching-and-solving activity, I raised the idea of using the variables arbitrarily. I wrote the equations $y = \sin(x)$ and $y = \arcsin(x)$ next to two unlabeled Cartesian coordinate systems. I asked the pair to describe how they would label each coordinate system for the given equation and what quantity each variable would represent in each case. Katlyn stated she would use the conventional x-horizontal, y-vertical axis labeling and that for the $y = \sin(x)$, x would represent angle measure but in $y = \arcsin(x)$, y would represent angle measure. Arya questioned, "Why do we do that?… It doesn't make sense… Just because we want this to be our input [*pointing to x in the equation* $y = x^{1/3}$] and that to be our output [*pointing to y in* $y = x^{1/3}$]? I feel like that's really the only…It's [*referring to switching the variables*] just so you can call your input x and your output y."

Although Arya identified that switching-and-solving maintained calling the input quantity x, this did not resolve her perturbation. For instance, Arya leveraged reasoning bidirectionally to question the need of a second graph if a relation (or function) and its inverse were meant to represent the same relationship between quantities, saying, "If they gave me this graph [*Graph 1*] and wanted me to find the information, with this [*the vertical axis*] as the input I certainly could… I could turn my head and look at it [*Graph 1 with the vertical axis as input*] and understand what that means. There's still no reason for this [*pointing to Graph 2*] graph. So… like when you switch then you're saying something new." Arya maintained that her switching-and-solving technique resulted in 'something new', which was incompatible with her anticipation that a function or relation and its inverse maintain the same relationship.

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Through much of the remainder of the last two teaching episodes, Arya experienced a state of perturbation as she attempted to relate her quantitative meaning for inverse to her switching-and-solving activity. By the end of the fourth teaching episode, Arya understood using variables arbitrarily (e.g., switching the quantitative referents of variables) to relate these meanings but continued to question why this would be done if a function and its inverse were meant to represent the same relationship. Arya's activity in contextualized situations, along with her reasoning bidirectionally with respect to the axes and her reasoning about variables as arbitrary, supported her in reorganizing her meanings for inverse relations (and functions). In later interactions, Arya maintained that a relation (regardless of if the relation was a function) and its inverse represented the same relationship and that in order to make sense of switching-and-solving she had to use the variables arbitrarily to represent the quantities under consideration (although she continued to question why she was taught to switch the variables).

Discussion and Concluding Remarks

By the end of the teaching experiment Arya understood that a relation and its inverse (regardless whether the relation was a function in the formal sense) represented the same relationship. However, developing this understanding was not trivial; it required Arya to reorganize her meanings for interpreting graphs (e.g., which axis could represent the input quantity), her meanings for variables (e.g., changing the quantitative meaning of x depending on the function under consideration), and her inverse function meanings (e.g., a function and its inverse represent something different). Whereas previous researchers have focused on students' and teachers' developing inverse function meanings via their understanding of function composition, these results provide insights into a different approach to inverse functions (or relations). To develop an understanding of a relation and its inverse as representing the same invariant relationship, the student in this study had to consider and reflect on various meanings she maintained (e.g., graphical conventions, interpreting variables, inverse procedures). As she reflected on her activity including her coordinating relationships between quantities, the student reorganized her meanings for various mathematical ideas so that she could adequately address the prompts in the *Graphing sine/arcsine task* (as well as all previous problems she was able to address). Previous researchers have indicated students can leverage quantitative reasoning to develop foundational meanings for various mathematical topics (Ellis et al., 2012; Johnson, 2012), and these results indicate students can reorganize already developed meanings via quantitative reasoning, although this process is not trivial.

In this report, I focused on Arya's development of inverse relation (or function) meanings via her reasoning quantitatively about relationships. Arya's activity had the potential to influence other meanings as well, including her function meanings. Future researchers may be interested in exploring how quantitative and/or bidirectional reasoning has the potential to support students in developing foundational function meanings. Additionally, this work examines the activity of a student who had already developed meanings for inverse function. Researchers may be interested in exploring how students who have not had formal instruction in functions and inverse functions (e.g., middle school students) could develop meanings for (inverse) function via their reasoning about quantities as these results indicate that such an approach has the potential to support students in developing productive inverse meanings.

Endnote

[1]Arya did discuss restrictions to the graphs in Figure 3(a) such that both Graph 1 and Graph 2 would represent functions regardless of chosen input quantity, but she typically worked with the arcsine relation (a multi-valued function).

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