TEACHERS' UNDERSTANDING OF RATIOS AND THEIR CONNECTIONS TO FRACTIONS

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In this study, we considered how middle school teachers understood the relationship between fractions and ratios. We used two instruments to collect data from 11 teachers and relied on Knowledge in Pieces as a lens for considering understandings teachers have and how coherent those understandings are. From our analysis, we developed three main findings: participants did not have a single definition for ratios; they used specific vocabulary when discussing ratios; and their language evoked additive strategies rather than multiplicative relationships. Further, we concluded that they each had a number of knowledge resources, but that those resources may not yet be well-connected to each other. This has implications for professional development.

Keywords: Teacher Knowledge; Middle School Education; Rational Numbers

Purpose & Background

In middle school mathematics, teachers are asked to teach an array of concepts for which they may have only limited understanding. One such area, proportional reasoning, has increased in prominence and emphasis by being considered its own content domain in the Common Core State Standards for Mathematics (National Governors Association & Council of Chief State School Officers, 2010). Despite the importance and richness of the proportional reasoning domain there has been a disproportionate focus on it in research (Lamon, 2007). The limited research available on teacher knowledge of proportions indicates that like students, teachers struggle with proportional reasoning (e.g., Akar, 2010; Harel & Behr, 1995; Orrill & Brown, 2012; Orrill & Kittleson, in press; Orrill, Izsák, Cohen, Templin & Lobato 2010; Post, Harel, Behr, & Lesh, 1988; Riley, 2010). Strikingly, in one study, Post, Harel, Behr, & Lesh (1988) found that their sample of teachers in grades 4-6 were unable to correctly respond to ratio and proportion items developed for students in those grades. In fact, on the ratio items, the 167 respondents answered less than 50% of the items correctly.

Lamon (2007) explained that proportional reasoning is one of "the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites" (p. 629). Despite this there is little research on teachers' understandings of proportional reasoning (Ben-Chaim, Keret, & Ilany, 2007; Lamon, 2007; Lobato, Orrill, Druken, & Jacobsom, 2011). Existing research suggests that proportional reasoning is conceptually difficult for teachers. This is, in part, because it is possible to rely on rote algorithms such as cross multiplication to get correct answers while overlooking the multiplicative nature of the relationship (Berk, Taber, Gorowara & Poetzl, 2009; Lobato et al., 2011; Modestou & Gagatsis, 2010; Orrill & Burke, 2013). Researchers have also suggested that teachers hold naïve conceptions about proportions (Canada, Gilbert, & Adolphson, 2008; Lobato et al., 2011). For instance, Canada, Gilbert, and Adolphson (2008) found that in a sample of 75 pre-service teachers only 28 were able to reasonably interpret a unit rate (e.g., amount per dollar) as useful for determining which package was a better buy when comparing two different size packages of ice cream.

Past research indicates an important link between the amount of knowledge a teacher demonstrates and its organization (Bédard & Chi, 1992; Ma, 1999; Orrill & Shaffer, 2012). For instance, Orrill and Shaffer (2012) found that the least expert teacher in their study demonstrated many ideas about ratios and fractions that were not interconnected while the most expert teacher introduced many ideas that co-occurred more frequently, suggesting stronger connections between them. We hypothesize that these stronger connections are an indicator of greater coherence. This finding is consistent with research in cognitive psychology that suggests expertise requires both an accumulation of knowledge and organization of that knowledge (Bédard & Chi, 1992). It is also consistent with seminal work in mathematics education such as Ma's study (1999) that showed that teachers with more connections between their mathematical knowledge resources were better able to interpret a variety of mathematical situations. As highlighted by Thompson, Carlson, and Silverman (2007), teachers with incoherent understanding can only teach disconnected facts. In contrast, a teacher with coherent understanding has the potential to support students in developing coherent understandings. Thus, coherence is a salient aspect of teacher knowledge (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Kaasila, Pehkonen, & Hellinen, 2010; Ma, 1999).

Two concepts that are important to proportional reasoning are fractions and ratios. Past research has found that the relationship between these two important concepts is not always clear (Clark, Berenson & Cavey, 2003; Sowder, Philipp, Armstrong, & Schappelle, 1998). This may, in part, be because of the organization of textbooks that frequently provide limited guidance on the definition of ratios and fractions and that deal with multiplicative structures in discrete unconnected ways, such that topics like the relationship between ratios and fractions are not shown or discussed (Clark et al., 2003; Sowder et al., 1998). These issues suggest that it is entirely plausible that teachers hold multiple knowledge resources about the relationship between fractions and ratios that are not coherently organized.

This study contributes to the growing knowledge base focused on teacher understanding by considering one aspect of proportional reasoning: relationships between fractions and ratios (e.g., Lobato & Ellis, 2010). Specifically, we considered the following questions: (1) how do 11 middle school teachers understand the relationship between ratios and fractions; and (2) how coherent are their understandings?

Theoretical Framework

Coherent & Robust Understandings

A coherent and robust understanding of ratios for middle school teachers must go beyond that of their students (Clark et al., 2003; Lobato & Ellis, 2010). Teachers need to understand that a ratio is a comparison of two quantities, where quantity is defined as "a measurable quality of an object- whether that quality is actually quantified or not" (Lamon, 2007, p. 630). A teachers' understanding of ratio should go beyond ways to express it, to include the understanding that a ratio is a multiplicative comparison and not an additive comparison (Lamon, 2007; Lobato & Ellis, 2010; Sowder et al., 1998). This is a critical understanding as the concept of ratio is considered crucial for the transition from additive to multiplicative reasoning (Sowder et al., 1998). Teachers need to be able to discern whether students are using additive or multiplicative reasoning (Sowder et al., 1998).

It is also important for teachers to understand the relationship between ratios and fractions. A fraction is more than simply a part-whole relationship. Fractions can be interpreted as a part-whole comparison, measure, operator, quotient, and ratio (Lamon, 2007). A common notion that students have is that all ratios are fractions – which is a limited conception given that ratios can be part-part relationships and given that a ratio is a comparison of two quantities, thus not a value that can be placed on a number line (Clark et al., 2003; Lamon, 2007; Lobato & Ellis, 2010). Teachers should both understand the relationship between fractions and ratios and have the ability to identify students' limited understandings to justify or refute them (Lobato& Ellis, 2010). And, teachers need to know that in many cases ratios can be meaningfully reinterpreted as fractions (Lobato & Ellis, 2010). For instance, in a salad dressing that is 2 parts of vinegar and 5 parts of oil, the ratio 2:5 expresses not only the part-part comparison, but also the multiplicative relationship—that there is 2/5 as much vinegar as oil.

Knowledge in Pieces

We rely on the Knowledge in Pieces theory (KiP; diSessa, 2006) for this study. KiP asserts that individuals hold understandings of various grain sizes that are used as knowledge resources in a given situation (Orrill & Burke; 2013). For novices, these knowledge resources are not well-connected to each other. As expertise develops, interconnections allow more knowledge resources to be invoked in appropriate situations. KiP offers a unique lens for exploring the development of expertise, which is dependent on the extent of the *coherency of knowledge* (Orrill & Burke; 2013). By coherency of knowledge we refer to multiple knowledge resources that are connected in robust ways allowing for *in situ* access to the resources. Coherence, combined with a robust set of knowledge resources, allows teachers to deal with complex situations in more efficient ways. This is consistent with previous research on expertise (e.g., Bédard & Chi, 1992), and Ma's (1999) concept of profound understandings of mathematics. We hypothesize that as a teacher develops coherence among knowledge resources, the teacher will be more fluent at teaching and doing mathematics.

KiP represents a departure from the deficiency model traditionally used in the study of teachers' knowledge. Much prior research has focused on what knowledge teachers do not "have" and the misconceptions that they do display. In contrast, KiP assumes that teachers have a wide variety of knowledge resources available to them, but that those resources may not be well connected. KiP also allows for identification of additional resources that could be important for a teacher to develop.

Methods & Data Sources

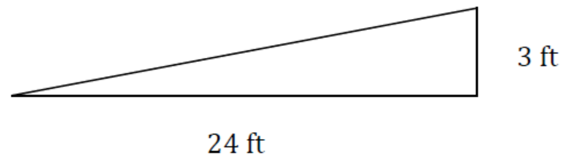
The participants were 11 middle school teachers (6 females) ranging from 1 to 18 years of experience from multiple schools within a single state. Data were collected from two interviews. One, the LiveScribe interview, was a paper-based protocol with think-aloud prompts that included 23 items. We mailed the interview protocol to the participants along with a LiveScribe pen, which recorded their spoken words as well as marks they made on their paper. The second source of data was from a 90-minute videotaped clinical interview including 18 items conducted with each participant. All recordings were transcribed verbatim to capture the knowledge resources evoked by the participants. We analyzed the data by focusing on the knowledge resources the teachers demonstrated on these tasks (not those resources they did not use). We used open coding (Corbin & Strauss, 2007) to identify codes for knowledge resources.

Interview Tasks

For the current study, we considered participants' responses to four tasks that focused specifically on the relationship of fractions and ratios. The Triangle task (Table 2), from the LiveScribe interview, explored teachers' understandings of the multiplicative relationship between the two sides of the triangle. All the other tasks were drawn from the clinical interview. Tasks 2 and 3 focused on situation related to salad dressing shown in Table 2. In Task 2, participants are asked to respond to one teacher's approach to making sense of the situation using an algorithm to find equivalent fractions. We then asked the teachers, "What does 2:5 mean as two-fifths? What is there

2/5 of in this situation?" Task 3 asked the participants to react to other teachers' responses to Task 2 as shown in Table 2. We considered only the second bullet point, "fractions and ratios are the same

Table 2: Interview Tasks

Task 1 Triangle Task
Some students in Mr. Warren's class have noticed that the ratio of 3 feet to 24 feet simplifies to 1 to 8. They also know that this ratio can be written as 1/8. However, they get confused about what the fraction 1/8 means in this situation.  a) What does the 1/8 mean in this situation? b) How would you explain that to your students?
Task 2? Oil & Vinegar Situation
Alexi made a batch of salad dressing using 2 tablespoons of vinegar and 5 tablespoons of oil. She would like to make a much larger batch that preserves the ratio of vinegar to oil. If she uses 15 tablespoons of oil, how much vinegar should she use?
Task 3 Teachers' responses to the oil & vinegar situation
• "I know that 6 is $\frac{2}{5}$ of 15. So I guess there's a two-fifth there." • "Fractions and ratios are the same thing." • "$\frac{2}{5}$ is a ratio here not a fraction. A fraction is a part-whole relationship like 2 T vinegar to 7 T of salad dressing, which is $\frac{2}{7}$ not $\frac{2}{5}$." • "I wonder if it has something to do with finding how much vinegar I would need for 1 part oil or how much oil for 1 part vinegar?"

thing" for this analysis. Finally, Task 4 asked each participant whether they believe fractions and ratios are the same. (Note: there are 4 participants who did not respond to Tasks 2 and 3 due to time constraints in the clinical interview).

Results

In our analysis, we found three main results related to our questions of how 11 middle school teachers understand the relationship between ratios and fractions; and how coherent those understandings are. First, the participants did not share a unified definition of ratios. Second, these participants used specific vocabulary to discuss ratios that differed from their fraction vocabulary. Finally, these participants relied on language that evoked a build-up strategy rather than language that suggests multiplicative relationships.

Multiple Definitions

Consistent with previous research (e.g., Clark et al., 2003), these participants seemed to draw from multiple knowledge resources in defining the relationship between ratios and fractions. The knowledge resources we saw among the 11 participants were comparison of two concepts, part-part and part-whole relationships, context as differentiating, and equivalence. For example, five participants focused on the similarity between the representations of fractions and ratios in talking about the relationship of the two concepts. For instance, Greg said in response to Task 3,

> Fractions and ratios are the same thing…I mean, a ratio can be written as a fraction, but again, you could write this as 2/7ths. … when you're thinking about it as a ratio it's important to define what you're comparing what the numerator and denominator are.

Mike and Alan also mentioned the idea of "comparing two things to each other" (Mike).

Nine participants relied on discussions of part-part and part-whole relationships. Bridgett and Alan both relied on the idea that ratios are part-part whereas fractions are part-whole, without clarifying whether ratios could be part-whole. For example, Bridgett explained, "When we first introduce ratios we say it's a part over a part and then we say for fractions it's a part over a whole." Allison, Ella, Mike, Larissa, and Greg added that ratios can be part-whole relationships. For instance, Allison explained that ratio is part-part but "sometimes it can be a part to whole relationship" as in the relationship of "the vinegar to the whole recipe" in Task 2.

A third set of knowledge resources considered context as differentiating ratios and fractions. David and Greg both discussed the need for using units (labels) with ratios. David asserted that fractions and ratios are the same except, "… with a fraction you don't need a unit. A ratio you should have some type of unit… you don't just put numbers." Larissa considered the need for context through word problems as a differentiating characteristic. She stated, "When I'm dealing with fractions I don't necessarily see it as a ratio unless it's in a word problem form."

Equivalence was the final knowledge resource on which participants drew. Greg, David and Ella explained that the 1/8 in Task 1 is a ratio rather than a fraction by referring to equivalent ratios. For instance, Greg explained that 1/8 is "the ratio between the two legs" and that we also could have "1-to-8, 2-to-16, 30-to-240, and those would still have the same ratio." Larissa, Ella, Greg and Allison relied on the use of similar triangles to demonstrate that all triangles similar to the given one have the same ratio as the original triangle.

Ratio Language

For these 11 participants, ratios evoked certain phrases. Most common among these was the phrase "for every", which was used by nine of the participants. For example, in describing the relationship in the Task 1 David said, "For every one foot on the short side of this triangle you have eight feet on the long side of this triangle." Similar language was used by six participants and two others used this language in Task 2.

We also noted that many participants used "*a* to *b*" when describing ratios versus "*a* out of *b*" or "*a*-*b*ths" language when describing fractions. For example, Ella justified her assertion that there is not a two-fifths in the oil and vinegar situation saying, "the two-fifths is not like two to five… like it is just fundamentally part to part. Pretty much all ratios are." Care in using differentiating language use was not consistent for all the participants.

Build-up Language

Our third main finding focused on the language selected by the participants. There was pervasive use of language that suggested additive reasoning. In particular, we found that10 of the 11 teachers used some variation of "for every" in their response. For example, when responding to part B of Task 1, Bridgette stated, "so every time you go up one you should go out eight points." This suggested a build-up strategy in that every time you add one to the short side, you add 8 to the long side. Another suggestion of additive reasoning came in statements of uncertainty about multiplication versus addition. For example, Allison noted, "for every one unit on one side the other side has eight times that unit. I almost said eight plus, but then that wouldn't work if it was eight plus so it has to be eight times that unit." This suggested a tie to addition for this participant. Only Autumn avoided use of this language in her responses.

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Conclusions

We examined 11 teachers' understandings of the relationship between fractions and ratios and how coherent those understandings were. The lack of a dominant focus for ratios and fractions suggests that students may be hearing a number of different definitions from their teacher. This is consistent with previous research and could partially be attributed to a wide array of definitions presented in textbooks (Clark et al., 2003) as well as to a lack of a single definition of these concepts in the field (Lamon, 2007; Lobato & Ellis, 2010; Vergnaud, 1988). For our research, the use of these resources raise questions about the coherence of teachers' knowledge. Holding many definitions that do not seem well-connected could suggest knowledge structures that are not robust enough to support an array of student thinking. For instance, if these teachers tell students that ratios are part-part relationships whereas fractions are part-whole relationships students may infer that ratio cannot be part-whole. Teachers with coherent and accurate resources for the relationship of ratios and fractions may be better able to support students in developing coherent understanding of these concepts.

We saw that language and context both seem to be important in considering knowledge resources for ratios and fractions. Many of the participants relied on certain phrases when discussing ratios. The participants were not consistent with these phrases and some used them interchangeably, which obscures the coherence or lack thereof of the concepts. Thus, language and context seem critical for the development of coherency of knowledge in this domain.

We found that several important aspects of a coherent and robust understanding of ratios and fractions were not evoked by the teachers. For example, not all described ratios as a comparison of two quantities. Also only five participants were able to reason about the relative value of one quantity to the other and six participants were unable to reinterpret a ratio as a fraction in Task 1. These teachers seemed to have access to knowledge resources for fractions and ratios, but relied only on ratio understanding in some cases.

This study considers areas in which participants may lack coherence in their understandings. For example, part-whole discussions only happened in the context of the oil and vinegar situation. In contrast, build-up strategies, which are more elementary (Lamon, 2007), were found across the tasks. For a coherent understanding, we would expect to see strong connections between consistently used knowledge resources comprising a robust understanding of ratios.

Scholarly Significance

Teachers need robust understandings of mathematics to support students' learning (e.g., Baumert et al., 2010). However, little research has been done on teachers' understandings of proportional reasoning to uncover how they conceptualize the relationship between fractions and ratios. Knowing how teachers understand the mathematics they teach has practical implications for guiding the development of effective support opportunities for teachers.

The teachers in our study have access to a variety of knowledge resources for fractions and ratios, but they have not necessarily developed coherent connections between those resources. Returning to the idea that expertise refers to having more structured knowledge (Bédard & Chi, 1992), this work unveils some possible connections between knowledge resources that teachers rely on when differentiating between ratios and fractions. More research needs to be done to highlight the kind of knowledge and the organization of the knowledge needed for teaching ratios and fractions.

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