DYNAMIC GEOMETRY SOFTWARE AND TRACING TANGENTS IN THE CONTEXT OF THE MEAN VALUE THEOREM

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In this paper we analyze and discuss the postgraduate students' performance related to the tracing of tangent lines to the curve of a quadratic function within Dynamic Geometry Software in the context of Mean Value Theorem. The purpose is to show the possibility of using Dynamic Geometry in promoting learning of such Theorem, based on its geometric interpretation. The theoretical elements adopted in this study are based on the instrumental approach to tool use. The results illustrate the epistemic role of the Dynamic Geometry Technique, as well as the difficulties associated with their paper-and-pencil Techniques.

Keywords: Technology; Advanced Mathematical Thinking; Teacher Education-Inservice

Background

In literature, there is evidence about the influence of using Dynamic Geometry Software (DGS) to encourage students' mathematical thinking (e.g., Guven, 2008; Leung, Chan & Lopez-Real, 2006; Reyes & Santos, 2009). In these studies the possibility of using dynamic geometry is raised to discuss mathematical relationships exploring different cases. In particular, Guven (2008) and Reyes and Santos (2009) show how the dragging and the locus, that emerge in the explorations that let the DGS, promotes the development of conjectures about the mathematical relationships of the objects embedded in the mathematical dynamic model, in the sense of Reyes and Santos. From these studies, and the interest of the community in analyzing the influence of the technological environments in teaching and learning calculus (Ferrara, Pratt & Robutti, 2006), we propose the possibility of using DGS to analyze its potential in promoting the learning of Mean Value Theorem (MVT), based on its geometric interpretation.

In Ferrara, Pratt and Robutti (2006) a study compilation is included about central concepts of calculus such as function, limit, derivative and integral in technological environments; from these backgrounds, we consider that it is important to research the role of technology in the learning of calculus in which the concept of derivate is embedded. Therefore, this paper focuses on studying the use of DGS in learning the MVT through its dynamic modeling.

The understanding of MVT is important because it is the base of contents like the criteria for maximums and minimums. Taking into account the use of DGS allows us to approach the MVT and the mathematical concepts associated, through mathematical dynamic models, and not just in analytic ways, as it is usually presented in textbooks. In this sense, in the dynamic geometry environments, the approach to MVT can be done by tracing tangent lines to the curve of a function in the context of its geometrical interpretation. Thus, the aim of this study is to answer the question: how does the use of dynamic geometry influence the tracing of tangents to the curve of a particular function in the context of the MVT, based on its geometric interpretation?

Theoretical Framework

The theoretical framework adopted in our study is *the instrumental approach to tool use* (Artigue, 2002; Lagrange, 2003, 2005). The use of this approach in dynamic geometry environments is feasible (Leung, Chan & Lopez-Real, 2006). According to Artigue (2002) the instrumental approach encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). In this sense, there are two possible

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directions within the instrumental approach: one in line with the cognitive ergonomics framework, and the other in line with the anthropological theory of didactics (Monaghan, 2007). In the former, the focus is the development of mental schemes within the process of instrumental genesis (Drijvers & Trouche, 2008). Within this direction, an essential point is the distinction between artifact and instrument.

In line with Chevallard's theory, researchers such as Artigue (2002) and Lagrange (2003, 2005) focus on the techniques that students develop while using technology. According to Chevallard (1999), mathematical objects emerge in a system of practices (*praxeologies*) that are characterized by four components: *task*, in which the object is embedded (and expressed in terms of verbs); *technique*, used to solve the task; *technology*, the discourse that explains and justifies the technique; and *theory*, the discourse that provides the structural basis for the technology.

Artigue (2002) and her colleagues have reduced Chevallard's four components to three: *Task*, *Technique* and *Theory*. The term *Theory* combines Chevallards's technology and theory components. The *Technique* is a complex assembly of reasoning and routine work and has both pragmatic and epistemic values; techniques are most often perceived and evaluated in terms of its pragmatic value, but their epistemic value contribute to the understanding of the objects they involve, that is to say, they are a source of questions about mathematical knowledge (Artigue, 2002, p. 248). According to Lagrange (2003), *Technique* is a way of doing a *Task* and it plays a pragmatic role (in the sense of accomplishing the task) and an epistemic role in that it contributes to an understanding of the mathematical object that it handles during its elaboration; it also promotes conceptual reflection when the technique is compared with other techniques and when discussed with regard to consistency. The consistency and effectiveness of a *Technique*, according to Lagrange (2005) are discussed in a theoretical level; mathematical concepts and properties and a specific language appear.

Our study is in line with the anthropological theory of didactics; thus the focus of this research is the epistemic value of technique. That is, we are interested in studying the students' Techniques they develop within the dynamic geometry environment.

The Study

In the present paper, we discuss and report the results of the designed Activity. Its rationale, the population and the data collection, is detailed bellow.

The Design of the Activity

The design took into consideration the Anthropological line of the instrumental approach. Thus, the three elements *Task, Technique and Theory* were used. The Activity, as Kiernan and Saldanha (2008) note, is a set of questions related to a central Task. In our case, the *Task* is "Drawing a tangent line to the curve of the quadratic function $f(x) = -(x - 3)^2 + 4$ and parallel to a secant line to the curve. The Activity consisted of two phases; the first one involves just working with paper-and-pencil, in order to know the techniques used by the participants in this environment. The second one includes working with the DGS, in order to know how the use of DGS influences and modifies the initial participants' techniques and what other emerges; both phases include technical and theoretical questions. The DGS used was GeoGebra.

The Task consists in given the function $f(x) = -(x - 3)^2 + 4$, participants are asked to plot the curve and draw a secant line to the curve and determine its equation (blue line, Figure 1). Once this part of the Task is completed, participants are asked a Theoretical question related to whether or not a tangent line (red line) to the curve and parallel to the secant line could be traced (i.e., the geometrical interpretation of MVT). If the answer is affirmative, they are asked to trace and determine its equation, first in a paper-and-pencil environment; then, using DGS (with the restriction that differential calculus techniques are not allowed in this environment). The Figure 1 shows a graphic representation of the proposed Task.

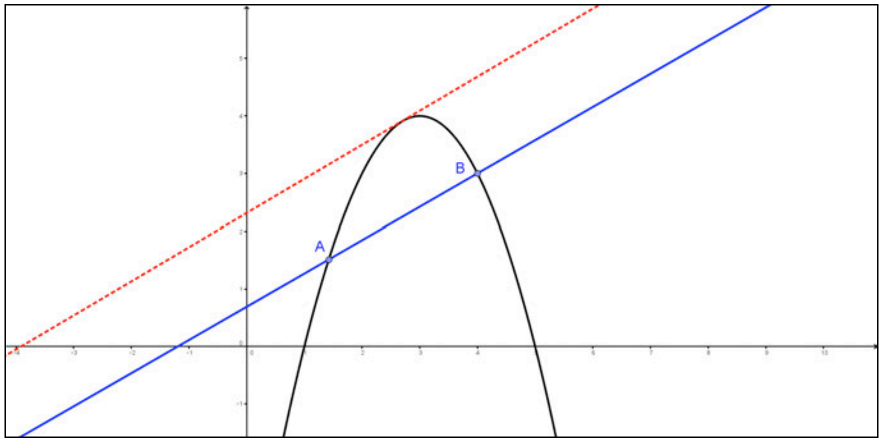


Figure 1: Graphic representation of the Task (graphic interpretation of MVT)

Population

The participants were 16 postgraduate students enrolled in a master program in the teaching of mathematics in Mexico. At the time of collecting data, they were in the 4th semester of the Master's degree. All participants knew GeoGebra and were familiar, at least a year and a half, with this software. All participants, except one, have teaching experience, some of them in university-level, others in senior-secondary-level and just a few in secondary-level. The professional degrees of the participants were among graduates in mathematics, engineers in different areas and one economist.

Implementation of the study

The data collection was carried out in three sessions, during one of the Master's degree courses conducted by one the researchers; each session lasted around 2 hours, which were recorded. The students worked in self-created pairs, in order to promote the dialogue among them and consequently make an audio recording about their own reflections in the use of GeoGebra according to the Task. Each team had a printed Activity, the GeoGebra software installed in their laptops, besides the SCREEN2EXE software which captures the computer screen in order to view the sequence of the students' work with the DGS. In this way, the research data sources include worksheets (printed Activity), the GeoGebra files, SCREEN2EXE files, video recorded files and the researcher's field notes.

Analysis and Discussion of Data

In this report we analyze and discuss the work of four pairs (Teams I, II, III and IV, henceforth), which exemplify the work done by all participants. The analysis conducted is of a qualitative nature inasmuch as we are interested in providing a detailed account of the kinds of Techniques that the participants used to solve the Task in both environments and the Theory they sustain. The analysis makes emphasis in the dynamic geometry techniques which were used by the students; that is to say, we are interested in research the kind of mathematical relations which they identified in the dynamic model of the Task that lead them to solve it.

On the paper-and-pencil work

The paper-and-pencil techniques and the Theory are based on the differential calculus. That is to say, on the usual procedure to determine a tangent line associating the function derivative with the slop of the tangents' family lines to the curve, although the kind of proposed function influenced in their reflections too. About the theory that sustains whether the possibility or not of drawing a tangent line with the required conditions, the participants refer the continuity of the function. Some of them describe it in an explicit way; others in the opposite way. The following Figures show the work of two Teams, that proves what is expressed, once they construct the graphic of the proposed function and pose the secant line equation to the referred function.

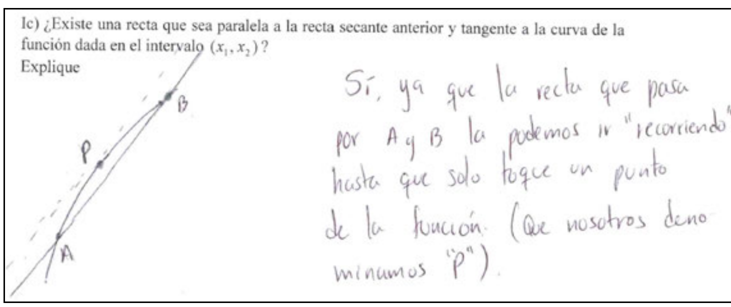
Ic) ¿Existe una recta que sea paralela a la recta secante anterior y tangente a la curva de la función dada en el intervalo (x_1, x_2) ?
Explique

1. Si, porque la función es continua en el intervalo.

Ic) Is there a parallel line to the secant and tangent to the curve of the given function in the interval (x_1, x_2)
Explain it
Yes, because the function is a continuous one in the interval.

Figure 2: Theory from Team I

To the question Ic) whether it will be possible to trace a line that it is parallel to the secant and tangent to the curve in a certain interval by the abscissas of the points where the secant line cuts the function of the graphic, the Team I sustains its Technique in the continuity concept of the function (Figure 2). Other Teams do not express their answers in an explicit way, for example, the Team II (see Figure 3). Note that when the students describe that it is possible dragging the secant line, based on a dynamic model of the Task, they demonstrate and idea of continuity of the function.



Yes, because we can dragg the line that passes through points A and B until it touches a single point of the function (which we denote by P).

Figure 3: Theory from Team II

Other teams justify their Techniques from their knowledge about the parabola; in particular about one of its specific points, the vertex. The Teams, which worked in this way, propose a parallel secant line to the axis of the abscissas, noticing the vertex of the parabola as the tangent point. However, they also showed ideas about the continuity of the function when they work in the dynamic geometry environment, as later discussed.

Because the participants' previous knowledge (Technique and Theory), it was expected that they would use differential calculus techniques to find and trace the tangent line equation. The analysis of the answers confirms this idea. Once the function is charted and the secant line equation to the curve is determined, those Teams that propose a secant non parallel to the axis of the abscissas use the derivate of the function to find the tangent point from solving the equation $f'(x) = m$, (where m is the slope of the secant line). The Figure 4 shows this Technique (*Derivate Technique*) from Team I.

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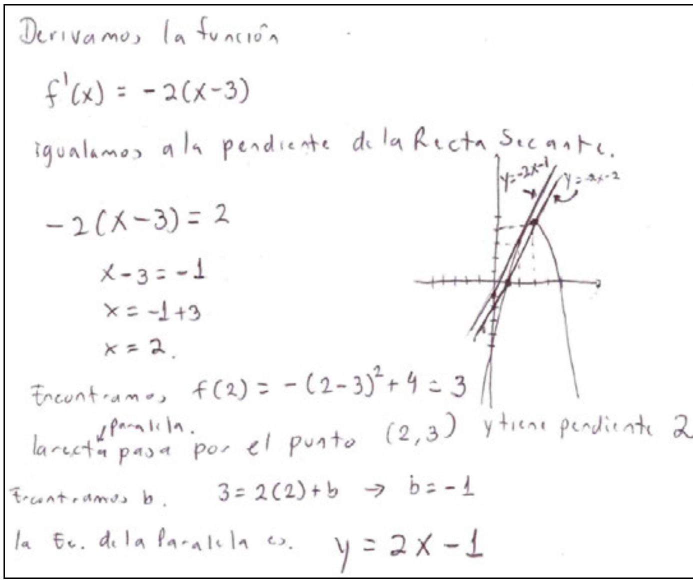

We derived the [given] function
$f'(x) = -2(x-3)$
We equate to the slope of the secant line
$-2(x-3) = 2$
.
.
.
$x = 2$
We calculate $f(2) = \ldots = 3$

The parallel line passes through the point (2,3) and has slope 2

We find b $3 = 2(2) + b \rightarrow b = -1$

The equation of the parallel line is $y = 2x - 1$

Figure 4: Technique from Team I

In other hand, the Teams that traced the parallel secant line to the axis of the abscissas used an Analytic Geometry Technique taking the maximum of the function like the tangent point. That is to say, they solve the Task using the equation $y = k$, where they identify k like the ordinate to the vertex with coordinates (h,k) of the parabola $y = a(x - h)^2 + k$.

On the Dynamic Geometry work

The work from the Teams in the GeoGebra environment, which was asked not to use differential calculus techniques, shows three characteristics. Some of them use *Algebraic Techniques* provoked by the dynamic model of the Task, and they use the DGS to trace their answers. Other Teams used the geometry dynamic characteristics and specific GeoGebra commands, which we called *Dynamic Geometry Techniques*, to model the Task and look for the mathematical relations that it involves. Others did not consider the use of DGS to explore mathematical relations, because it is taken as obvious. Next, we present examples from each one of these cases.

The Team III was one of those which traced the parallel secant line to the abscissas axis (as it is shown in Figure 5). On the offered explanations by one of the Team members (Student A) it is found that, for them, the answer to the Task is obvious, based on their work that they developed with paper-and-pencil. The following extract illustrates this case.

> Student A: What we did was tracing this [shows a point that was traced on the function of the graphic], and then, traced the parallel [to the secant] […] we moved it, moved it, moved it, move it [they dragged it] until the tangent point was found, which it is easy for us because it is the parabola vertex.

As it is shown in the transcription, the students used parallel and dragging commands as Technique. This let them trace a parallel to the secant line and that passes through a point (traced by them) over the graphic of the function. The mathematical relationship which is shown in their dynamic model is that the parallel line that passes through the parabola vertex fits with the given conditions of the Task. Nevertheless, the tangent point is known in advance, it is not a result from the explorations in the DGS. This is to say, their paper-and-pencil Technique and Theory (their knowledge about the parabola) let them solve the Task in the dynamic environment. The answer that

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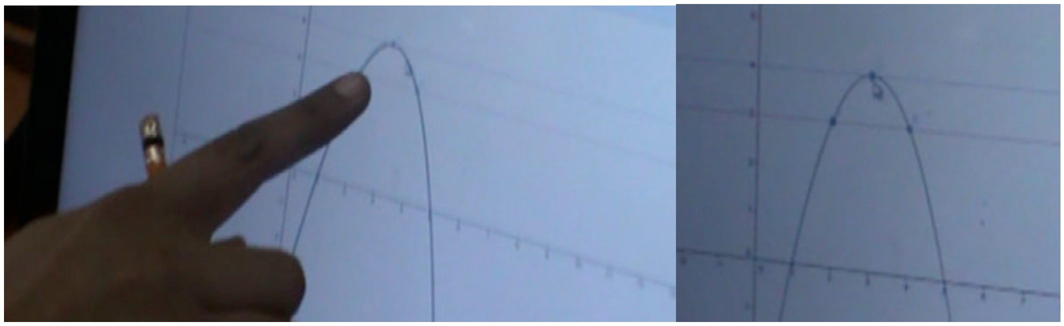


Figure 5: Dragging Technique from Team III

they give is particular, because if the secant line conditions are changed, the line which is proposed by them as the solution will not be the tangent, it just will keep the parallelism condition.

Meanwhile, for the Team I, the dynamic model leads them to try different paper-and-pencil techniques that differ from the calculus. They observed a mathematical relationship in the dynamic model of the Task which led them to the solution; it consisted of a system of equations with the parabola equation $(y = -(x - 3)^2 + 4)$ and the equation of the tangent line $(y = mx + b)$ so that the solution has multiplicity 2 (to fit with the tangent condition), where m is the value of the slope of the secant (in order to fulfill with the parallelism condition). In this way, they calculate the value of the parameter b (of the tangent line). Figure 6 shows this algebraic Technique.

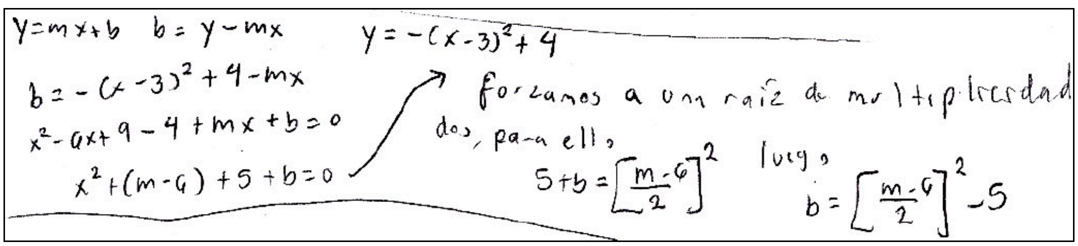
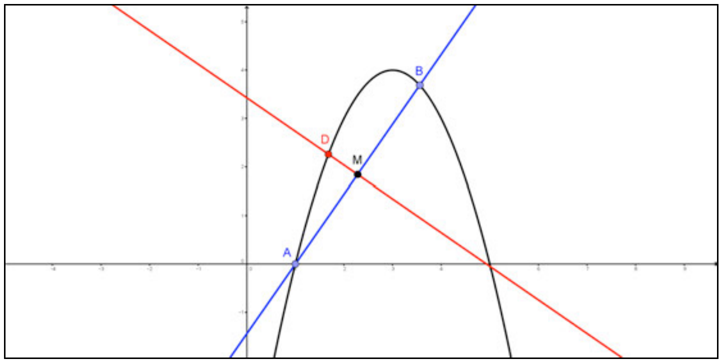


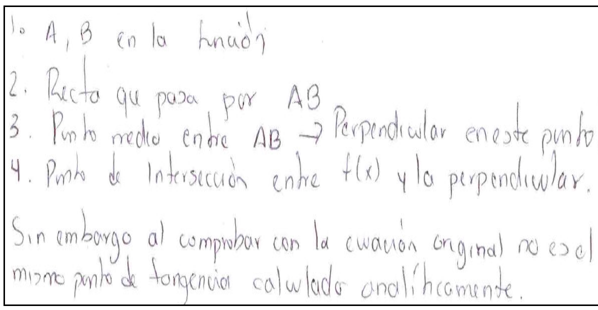
Figure 6: Algebraic Technique from the Team I

The given solution from the Team I is general; however, their answer is not supported by any Dynamic Geometry Technique, but it is an algebraic process. GeoGebra encouraged them to reflect in other alternative paper-with-pencil Technique. Once they found an expression for b in terms of m, they use GeoGebra to graph the equation with these parameters.

The Team II explored in the dynamic model by dragging through a slider as a possible Technique to solve the Task. Nevertheless, this technique lets them an approximation to the solution of the Task. This Team introduced in GeoGebra the requested equation $y = mx + b$ of the line, where m is the same as the value of the secant slope (with this, the condition of parallelism is completed) and associated the parameter b, the ordinate to the origin, with a slider; in order that they manipulate the slider (dragging the line) and propose a solution when, by trial and error, they observe in the graphic representation that the line fulfills the tangent conditions. However, they are aware, using the zoom Technique that their answer is just an approximation.

Finally, the Team IV tried to solve the Task, in the GeoGebra environment, based on the possibilities that the DGS offers. It is interesting how this Team, using the dynamic model, observes mathematical relations in the dynamic construction that they propose in GeoGebra (Figure 7). For this Team, the explorations they did in the DGS let them to conjecture that the middle point M from the segment AB (see Figure 7), the points where the secant (blue line of the Figure) crosses the curve; in particular, the perpendicular bisector (red line) of this segment, lead them to the find the tangent point.





1. *A, B on the function.*
2. *Line that passes through AB.*
3. *Midpoint point of AB segment, perpendicular [to the AB segment] in this point.*
4. *Intersection point between f(x) and the perpendicular.*

However, when comparing with the original equation, we found out that it is not the same tangent point calculated analytically.

Figure 7: Dynamic Geometry Technique from Team IV

In this case, the students established the tangent point as the intersection of the perpendicular bisector and the function of the curve (Point D in Figure 7). Nevertheless, they found out that this Technique do not lead them to the solution, making comparisons with their initial paper-and-pencil Technique. The most important thing about the work from Team IV is to realize that the dynamic model of the Task let them make conjectures about the midpoint of the segment AB, which, actually, it is related to the solution of the problem.

Conclusions

The offered examples shown in the previous section let us know the influence of the dynamic geometry software, from the participants' Technique and Theory, in the tangent traces. In the paper-and-pencil environment, two types of Techniques are identified, one based on the differential calculus knowledge; the other one, based on the analytic geometry knowledge. Regarding the Theory, it is related to the concept of continuity of the function.

The influence of dynamic geometry is shown in those Teams which use a dynamic model that do not use a parallel secant line to the abscissas axis. Therefore, in one side, the DGS encourages the emergence of paper-and-pencil Techniques based on the explorations in the dynamic model. In other hand, DGS lets them work with their own Techniques and commands from the dynamic geometry and establish relationships between the mathematical objects involved and the emergence of others (for example the perpendicular bisector). It also let them contrast paper-and-pencil Techniques with the software Techniques. According to the instrumental approach, this contrast of Techniques encourages to a reflection in a theoretical level.

In this way, the emergence of new paper-and-pencil Techniques and the reflections about the mathematical relationships provoked by the dynamic models in the use of the DGS show the epistemic role of the dynamic geometry Technique in the trace of tangent lines in the TVM context.

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In addition, the results show difficulties associated with their paper-and-pencil Techniques in the sense of holding to this environment, and not exploring the software potentials.

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