PRE-SERVICE TEACHERS' UNDERSTANDING OF FRACTION OPERATIONS: PROVIDING JUSTIFICATION FOR COMMON ALGORITHMS

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This study examined pre-service elementary teachers' change in their understanding of fraction operations while taking a mathematics methods course. Specifically, their explanations and justifications for common algorithms for multiplication and division of fractions were coded using an existing framework (SOLO; Biggs, 1999) for the assessment of understanding. Results indicated that most students made improvement in terms of their level of understanding around fraction algorithms. Implications for mathematics teacher educators are discussed.

Keywords: Elementary School Education; Teacher Education-Preservice; Mathematical Knowledge for Teaching; Reasoning and Proof

Currently, the amount of conceptual understanding pre-service elementary teachers hold around common algorithms is weak (Ball & Bass, 2002; Simon, 1993); however, a push for conceptual understanding is being deemed necessary for all students (CCSSO, 2010; NCTM, 2000; Stylianides, Stylianides, & Philippou, 2007). This means that pre-service elementary teachers need to hold a deeper understanding of common algorithms if their own elementary students are expected to understand the meaning behind each algorithm (Ball & Bass, 2002; Ball, Thames, & Phelps, 2008). Although previous studies have focused on elementary teachers' conceptual understandings (Simon & Blume, 1996; Yackel & Cobb, 1996) more work is needed around fraction algorithm understanding and how teachers justify those algorithms for their students. The current study aims to begin to fill that void.

The purpose of the study was to examine how pre-service elementary teachers provide explanation and justification for algorithms around fraction operations. Specifically, the two research questions were: (1) What are pre-service elementary teachers' levels of understanding of algorithms for the multiplication and division of fractions *before* and *after* experiencing instruction focused on fractions in a mathematics methods course?; and (2) How do pre-service elementary teachers *change* in their level of understanding of the algorithms?

Theoretical Framework and Related Literature

This study draws on the work of Skemp (1976) who defined the ideas of relational versus instrumental understanding. According to Skemp (1976), relational understanding is equivalent to conceptual understanding such that it is knowing *why* something happens, whereas instrumental understanding is similar to procedural understanding such that it is taking a rule and using it without understanding. In the past twenty years, there has been much attention to developing relational understanding in pre-service teachers (e.g., Ball, 1990; Ball & Bass, 2002; Simon, 1993;). Researchers have noted the specialized content knowledge that is specific to the work of teaching (Ball, Thames, & Phelps, 2008) and have emphasized that in order for teachers to be able to answer students' questions of "why," they must have a more robust relational understanding than that of their students.

One such study, performed by Eisenhart et al. (1993), followed Ms. Daniels, a student teacher who tried explaining the "invert and multiply" rule to a student in her class and abandoned the explanation midway through when she realized the example she was using pertained to multiplication, not division. Another study, performed by Ball (1990), gave the example of Allen, an

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elementary education major who also struggled with the “invert and multiply” rule for division of fractions and could not generate an example that did not reference multiplication. Much of the challenge for our work as mathematics teacher educators is to prepare pre-service teachers to answer these types of questions effectively and not give the reasoning “because it’s the rule”. To do so requires the development of a deep conceptual understanding of the mathematics for pre-service teachers.

To measure understanding in mathematics, researchers have typically used open-ended assessments and interviews that then must be analyzed. One analysis tool, and the tool used in this study, is the Structure of the Observed Learning Outcome Taxonomy (SOLO; Biggs, 1999) as displayed in Figure 1. The nature of the SOLO Taxonomy is one such that student learning is examined as they move from a lower level of understanding to a higher, more abstract, level of understanding. As students progress through the levels, they retain the traits from the previous level; in other words, each level builds on the previous. According to a study performed by Ball (1990), pre-service teachers’ mathematical understandings typically are found to be at the Unistructural level where they are simply reciting algorithms in their explanations to students.

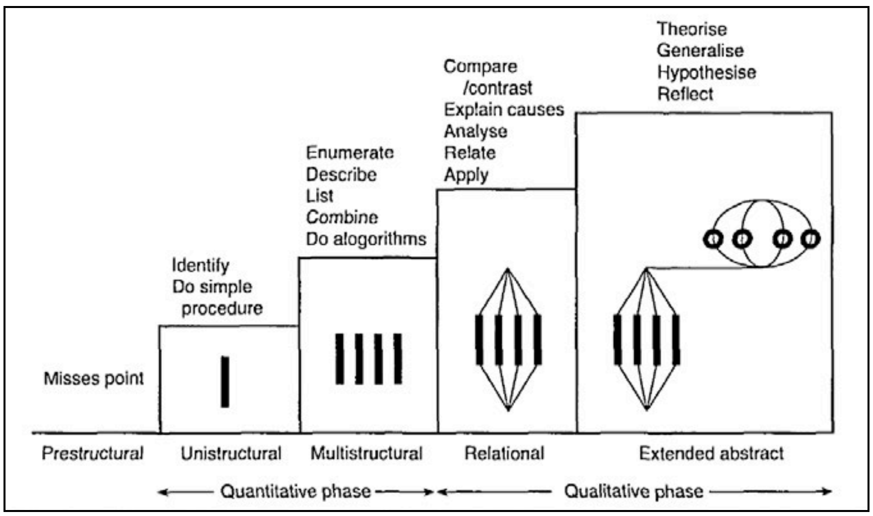

Figure 1: SOLO Taxonomy(Biggs, 1999, p. 67)

With the continued emphasis on conceptual understanding in documents such as the National Research Council’s *Adding it Up* (2001) and in the Common Core State Standards for Mathematics (2010), there is a need to better understand pre-service teachers’ conceptual understanding so we can better prepare them for their future work. Furthermore, historically, fractions have been difficult for both children and adults in the United States (Lamon, 2005). This speaks to the need for further research focused on fraction operations and namely pre-service teachers’ understanding of them, as is in the case in the current study.

Methods

Participants

The participants in this study were forty-eight juniors in an elementary education program who were enrolled in a mathematics methods course, the second in a two-course sequence and focused on multiplicative reasoning in grades 3-5. Data was collected from two sections of this course, which were taught by a total of four instructors, two teaching one section and the other two teaching the other section. All lessons were created as collaboration between the four instructors; therefore, all

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students experienced the same tasks and activities during the class sessions. Of the students enrolled in the methods course, 96% had taken one or two Calculus courses, either in high school or at the college level, or they had taken *Calculus for Elementary Teachers,* which focuses on a conceptual understanding of Calculus-related topics. The mathematical background of these students is important to note because the focus of this paper is on the pre-service teachers' conceptual understanding and level of justification they can provide to students. One can see from the level of mathematics achieved by these students that they have experienced the content they will be expected to teach; however, they may or may not be able to explain the ideas conceptually.

Intervention

After the pre-service teachers completed the pre-assessment, they participated in a five-week unit of instruction during which they learned about algorithms for fraction operations and also had opportunities to examine student work. Examples of tasks completed during the instruction period are shown in Figures 2 and 3.

Misconceptions with Fraction Multiplication

1. Maisy draws a picture like the one shown to depict $3 \cdot \frac{4}{5}$. Maisy concludes from her picture that

$$3 \cdot \frac{4}{5} = \frac{12}{15}$$

Because 12 pieces out of 15 are shaded. Is Maisy right? If not, where is her reasoning flawed?

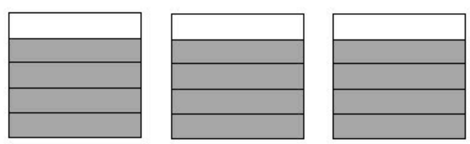


Figure 2: Misconceptions with Fraction Multiplication. Adapted from Sybilla Beckman "Mathematics for Elementary Teachers" fourth edition.

Popcorn Problem #1 **Name** ______________________________

At the Carolina Popcorn Shoppe, the managers direct their employees as to how much popcorn constitutes a serving when a customer orders one. There are currently 6 cups of cheese popcorn at the serving counter. How many servings can be made if the customer is given:

a. ½ cup of cheese popcorn?
b. ¼ cup of cheese popcorn?
c. 1/6 cup of cheese popcorn?
d. 1/3 cup of cheese popcorn?
e. 2/3 cup of cheese popcorn?
f. 3/4 cup of cheese popcorn?
g. 5/6 cup of cheese popcorn?

Solve this problem using pictures or another tool.

Can you write a mathematical equation to represent your model?

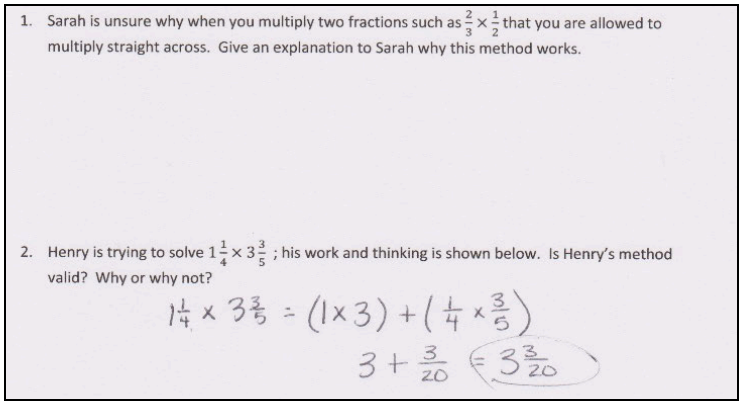
Describe the pattern that you notice, if any, across the sub-questions a-g.

Figure 3: Building Towards the Fraction Division Algorithm.

The activities were selected to build pre-service teachers' conceptual knowledge of fraction-based algorithms as well as their ability to recognize common student errors relating to fraction multiplication and division. Class readings were also given to help further pre-service teachers' understanding related to the given topics.

Measure

Before and after the instructional sequence, participants were asked to complete an assessment (see Figures 4 and 5) related to algorithms for fraction multiplication and division. Prior to instruction, the participants had not worked with multiplication and division of fractions in either of their methods courses.



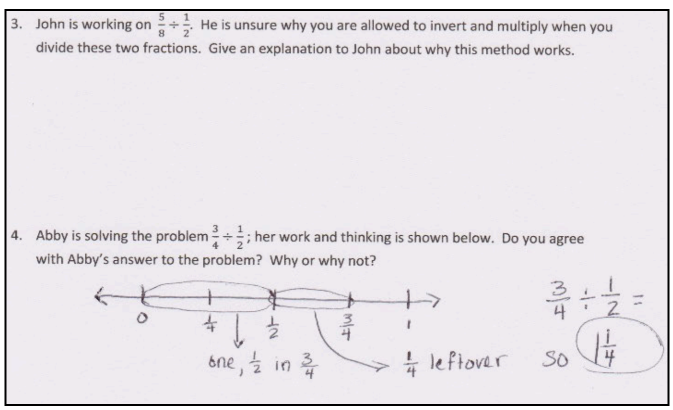
1. Sarah is unsure why when you multiply two fractions such as $\frac{2}{3} \times \frac{1}{2}$ that you are allowed to multiply straight across. Give an explanation to Sarah why this method works.

2. Henry is trying to solve $1\frac{1}{4} \times 3\frac{3}{5}$; his work and thinking is shown below. Is Henry's method valid? Why or why not?

$1\frac{1}{4} \times 3\frac{3}{5} = (1 \times 3) + (\frac{1}{4} \times \frac{3}{5})$

$3 + \frac{3}{20} = 3\frac{3}{20}$

Figure 4: Questions 1 & 2 from Pre/Post Assessment regarding fraction multiplication.



3. John is working on $\frac{5}{8} \div \frac{1}{2}$. He is unsure why you are allowed to invert and multiply when you divide these two fractions. Give an explanation to John about why this method works.

4. Abby is solving the problem $\frac{3}{4} \div \frac{1}{2}$; her work and thinking is shown below. Do you agree with Abby's answer to the problem? Why or why not?

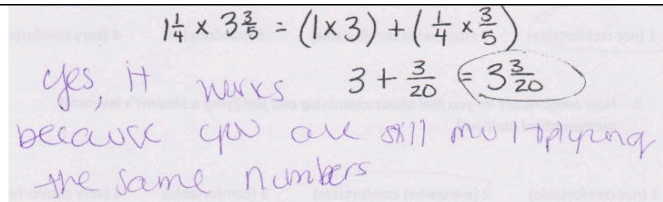
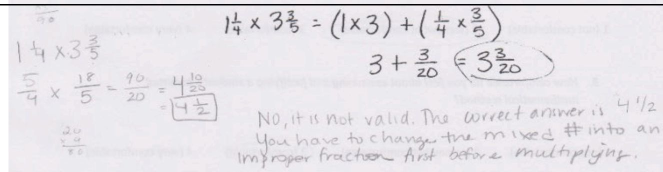
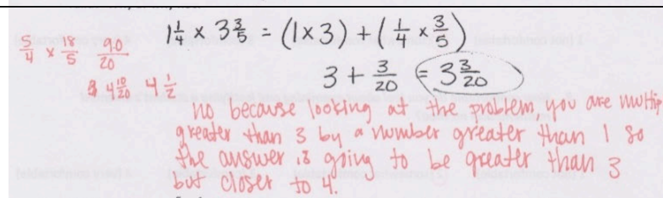
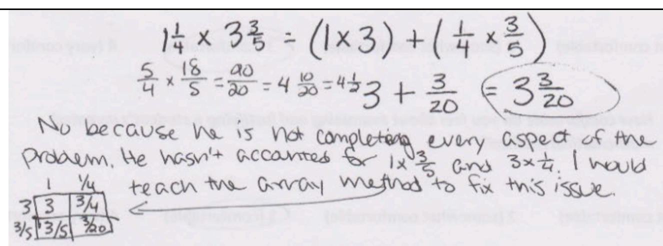
Figure 5: Questions 3 & 4 from Pre/Post Assessment regarding fraction division.

Analysis

Answers were coded according to the levels of the SOLO taxonomy with numbers from 0 to 4 assigned to each of the levels, with Preoperational being a 0 and Extended Abstract being a 4. Two independent raters coded 25% of the pre- and post-assessments in order to check for inter-rater consistency. 98% of the two raters' codes were either exact matches or within one scale point of each other (with 76% exact match agreement). Table 1 provides an example for each level in the SOLO taxonomy as well as a justification for the assigned code.

Beyond the example in Table 1, we now provide a general overview of the coding scheme. Responses coded as the Prestructural level meant the pre-service teacher misinterpreted the question or did not provide an answer. The Unistructural level meant the pre-service teacher did not explain, but instead recited the algorithm to the student as a means of justification. In terms of examining student work, they identified something was wrong, but could not pinpoint exactly where the mistake was occurring. The Multistructural level meant the pre-service teacher recited the algorithm, but gave a further justification for the specific example; however, the justification was not complete or was

Table 1: Examples of Student Responses for each Level.

SOLO Taxonomy Level	Example of Student Response	Justification for SOLO Level
Preoperational (0)		Incorrectly answered the problem.
Unistructural (1)		Recited the rule of turning the mixed numbers into improper fractions.
Multistructural (2)		Recited the rule, but began to display signs of conceptual understanding in the reasoning of estimating the answer.
Relational (3)		Understood the numbers were not fully decomposed and wanted to teach Henry how to view the problem in terms of an area (array) model.
Extended Abstract (4)	N/A – an example might include generalizing the problem to that which includes variables instead of numbers.	No examples of student work provided.

incorrect. In terms of examining student work, the pre-service teacher described what had been done and identified the student's mistake. Within the Relational level, the pre-service teacher provided an explanation in which the idea was fully explained conceptually in terms of the specific example. In regards to examining student work, they correctly identified the problem and explained what it meant in terms of the particular example. Finally, the highest level of justification was the Extended

Abstract in which the pre-service teacher gave a generalized proof in terms of variables, and explained the student's method in terms of a generalized approach.

Results

For question #1 on the assessment, the average score was a 0.50 on the pre-assessment and a .96 on the post-assessment (change of 0.46) on a scale of 0-4. Question #2, resulted in a 0.63 for the pre-assessment and a 1.15 for the post-assessment (change of 0.52) on a scale of 0-4. Question #3 resulted in a 0.40 for the pre-assessment and a 0.77 for the post-assessment (change of 0.37) on a scale of 0-4. Question #4 resulted in a 0.71 for the pre-assessment and a 1.46 for the post-assessment (change of 0.75) on a scale of 0-4. A total score for the entire pre-assessment was .56 and a 1.08 for the post-assessment. A paired t-test was performed for each question with 95% confidence, and all questions showed statistically significant improvement from pre to post assessment (all p-values < .05). A breakdown of the count of students for each level and question for the pre and post assessment is shown in Table 2:

Table 2: Count of Students in each Level and Question.

SOLO Taxonomy Level	Question #1		Question #2		Question #3		Question #4	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Preoperational (0)	34	20	19	13	30	28	25	19
Unistructural (1)	7	13	28	19	17	9	17	7
Multistructural (2)	4	12	1	12	1	5	1	3
Relational (3)	3	3	0	4	0	6	5	19
Extended Abstract (4)	0	0	0	0	0	0	0	0

As one can see from Table 2, no student was classified as Extended Abstract. This may have been because the course did not focus on formal proof and the questions were worded in such a way that they did not suggest giving a formal proof. Before instruction, students typically fell between Preoperational and Unistructural. In other words, pre-service teachers either did not know how to explain the problem to the student or they just simply recited the algorithm. After instruction, most students were between Unistructural and Multistructural; therefore, many students were still reciting an algorithm, but several were also making sense of the algorithm conceptually. The most improvement occurred on question #4 regarding Abby's invented solution to dividing fractions. Many students on the post-assessment recognized the answer of $1\frac{1}{4}$ was not correct because the leftover $\frac{1}{4}$ piece referred to a whole of 1 and not a whole of $\frac{1}{2}$. Therefore, the leftover $\frac{1}{4}$ piece was actually $\frac{1}{2}$ of the whole of $\frac{1}{2}$.

Discussion

In terms of the course, the students were not asked to work formal proofs, but instead use an example to explain a rule in mathematics; therefore, it was expected that no student would fall in the extended abstract category. Second, the questions were given to the student in terms of a particular example and they were asked to explain to the student why the algorithm worked. There might have

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been students who fell into the Extended Abstract category if the questions were reworded more generally, such as: "explain why the algorithm for multiplying fractions works".

Additionally, the results found from the study hold true with the study performed by Ball (1990), such that prospective teachers' "notions of mathematical explanation seemed to mean restating rules" (p. 138). Although there was a positive increase in levels, work needs to be done to better develop pre-service teachers' conceptual understanding. Finally, some of the increase in scores may have been attributed to fraction division being discussed during the same day of the post-assessment due to time restraints in the course. If the study were performed again, a period of time should be given between the instruction and the post-assessment to check for continued understanding beyond the day of instruction. .

Elementary teachers need to have a deep understanding of the material in order to "handle certain mathematical issues that may arise in the classroom and recognize rudimentary versions of mathematical [proof] in their students' arguments" (Stylianides et al., 2007, p. 148). By strengthening pre-service teachers' conceptual knowledge of mathematics and helping them form solid justifications as to why procedures work, the difficulties secondary students face when they are abruptly introduced to proof in the upper grades may be diminished (Stylianides, 2007). Therefore, in order to improve our education system, we need to improve teachers' knowledge of mathematical content as well as their overall concept of justification (Ball & Bass, 2002; Toluk-Uçar, 2009). The ways in which this can be achieved were highlighted throughout this paper such as: providing more courses in explanation and justification, spending more time on difficult topics, explaining in detail why algorithms work, and providing examples of teachers who teach conceptually through videos or classroom observations. Additionally, more courses involving various elementary mathematics topics are needed to improve pre-service teachers' conceptual understanding.

This paper only touched upon instruction given to students for a fraction-related unit. More research needs to be done to see how students provide justification for other mathematical algorithms, not just for fractions. Also, further research is needed to investigate whether higher-level mathematics courses are a factor in pre-service teachers' ability to justify solutions to students or if there are other factors involved. Finally, interviews of students who performed at the Relational level would be beneficial to see if those students could move into the Extended Abstract level with more time and guidance. Overall, instruction in this study was successful in helping to improve students' scores between levels of the SOLO taxonomy. Therefore, instruction seems to be one stepping stone in helping pre-service teachers in their endeavor to become proficient in explanation and justification.

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