PRESERVICE TEACHERS' FRACTIONAL CONCEPTS IN SOLVING ADVANCED FRACTION PROBLEMS

Mi Yeon Lee
Arizona State University
mlee115@asu.edu

Ji-Won Son
University at Buffalo
jiwonson@buffalo.edu

Talal Arabeyyat
University at Buffalo
talalhas@buffalo.edu

This study investigated how pre-service teachers' fractional concepts are related to solving problems involving advanced fractional knowledge. 96 Participants took a written test including three fractions questions about fraction concept, fraction comparison, and multiplicative relationship involving fraction quantities and composite units. The data were analyzed using an inductive content analysis approach. Findings suggest that many of our PSTs developed limited understanding of fraction sub-constructs and thereby were not able to solve the three problems. Another finding is that when PSTs relied on only one sub-construct such as part-whole, they tend to provide incorrect answers. This finding implies that PSTs' understanding of fractions as measure and operator may be a foundation of solving problems involving advanced fractional knowledge.

Keywords: Teacher Education-Preservice; Teacher Knowledge

Objectives or Purpose of the Study

The purpose of this study is to characterize profiles of the mathematical competence of pre-service teachers in the topic of fractions. Especially this study investigates how pre-service teachers' fractional concepts are related to solving problems involving advanced fractional knowledge. Since Shulman (1986) coined the notions of content knowledge and pedagogical content knowledge, many researchers have investigated what teachers know and how they know about mathematics and teaching mathematics over the past three decades. They reported teachers' insufficient knowledge about teaching mathematics, in particular, elementary in-service and pre-service teachers' lack of understanding about whole numbers, fractions, and fraction operations (e.g., Behr, Harel, Post & Lesh, 1993; Ma, 1999; Mack, 2001; Steffe, 2003).

This line of research studies suggests that teachers should develop a profound understanding of fundamental mathematics (Ma, 1999) and mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008), which encompasses knowledge that teachers use in teaching practice such as selecting and using effective representations, deftly assessing students' work, and providing appropriate remediation. In particular, teachers' capability to solve problems differently and to use different representations of mathematical ideas is considered to be an important aspect of mathematics education. Empson (2002) highlighted that "the key in fraction instruction is to pose tasks that elicit a variety of strategies and representations" (p. 39) and, that representational models used by teachers (e.g., pizzas, number lines, and fraction bars) engaged and facilitated students' learning of initial fraction knowledge.

However, despite a large number of research studies focusing on teachers' problem-solving, representational knowledge, and computational skills (e.g. Eisenhart, Borko, Brown, Underhill, Jones, & Agard, 1993), several questions still remain unanswered regarding preservice teachers' ability to solve problems, their ability to translate from one mode of knowledge to another, and possible difficulties preservice teachers have in connecting different modes of knowledge. Lester and Kehle (2003) highlighted that "far too little is known about problem solving" and in particular, about teachers' problem solving strategies and their abilities to transfer in problem solving (p. 510). This issue should be addressed in the teacher education of preservice teachers of mathematics at all levels.

As a way to characterize profiles of the mathematical competence of pre-service teachers in the topic of fractions, this study investigates how pre-service teachers' fractional concepts are related to solving problems involving advanced fractional knowledge. The research questions that guided the

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Articles published in the Proceedings are copyrighted by the authors.

study were: (1) How do PSTs solve fraction problems and what strategies do they use? (2) How does PSTs' conception of fractions relate to problem solving ability involving advanced fractional knowledge?

Theoretical Framework

Studies on Fractions (Five sub-constructs)

According to research, understanding fractions clearly indicates understanding five possible constructs that fractions can represent (Clarke, Roche, & Mitchell, 2008; Sibert & Caskin, 2006). Although the meaning of part-whole is dominantly used to represent fractions in mathematics textbooks, many mathematics researchers believe that students would understand fractions better when they are exposed to the other meanings of fractions (Clarke et al., 2008). The first construct is *part-whole*, which goes well with an example of shading parts out of a whole. The second construct is *measurement*. Measurement includes identifying a unit length and then iterating the unit length to determine the length of an object. The third construct is *division*. When considering a sharing context, people can connect division to fractions. For example, in the context of finding a person's share to fairly share 3 candy bars with 4 people, each person will receive ¾ of a candy bar. The fourth construct is *operator*. That is, fractions can be used to indicate an operation. For example, when John has $21 and Mary has 2/3 of John's money, Mary's money will be 14 dollars, which indicates 2/3 of 21 dollars. Students who represent "two-thirds the amount of 21 dollars as "2/3 x 21" may have developed a concept of a fraction as an operator. The last construct is *ratio*. That is, fractions can be used to represent part-part ratio or part-whole ratio. For example, when there are four red marbles and seven blue marbles, the ratio 4/7 could be used to indicate the ratio between red marbles (part) and blue marbles (part). Also, the ratio 4/11 could represent the ratio between red marbles (part) and total marbles (whole). In this study, we provided pre-service teachers with three fractions questions, which particularly used part-whole construct and operator construct. According to Usiskin (2007), operator construct is not stressed enough in school curricula although just knowing how to represent fractions using part-whole construct doesn't guarantee knowing how to operate with fractions in other areas of curriculum where fractions occur (Johanning, 2008).

Studies on Teacher Knowledge (MKT and CK)

Shulman (1986) put forward the idea that teachers have specialized knowledge of teaching, which is what differentiates a teacher from a subject matter specialist. This notion is referred to in his article as the distinction between *content knowledge, pedagogical content knowledge,* and *curricular knowledge. Content knowledge* is subject matter knowledge of mathematics. *Pedagogical content knowledge* is topic specific pedagogical knowledge needed to teach mathematics. *Curricular knowledge* refers to educational programs made for the teaching of specific subjects and topics at a given grade level, including instructional materials available in relevant programs, and affordances and constraints in the use of such specific instructional materials (Shulman, 1986). More recently, Ball and her colleagues (1990; 2008) proposed a new term, *Mathematical Knowledge for Teaching* (MKT), which is defined as "a practice-based content knowledge for teaching built on Shulman's (1986) notion of pedagogical content knowledge" (Ball, Thames, & Phelps, 2008, p. 389). Mathematical Knowledge for Teaching focuses on identifying students' mathematical thinking and on the understanding required to teach specific topics in mathematics. MKT distinguishes mathematical content knowledge into three strands: *common content knowledge*, *knowledge at the mathematical horizon*, and *specialized content knowledge*. C*ommon content knowledge* is described as the mathematical knowledge held in common with others who know and use mathematics in various professions or occupations, which is what Shulman meant by his original *content knowledge*. *Specialized content knowledge* is described as mathematical knowledge about the ways that mathematics is taught to students by a teacher. *Knowledge*

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Articles published in the Proceedings are copyrighted by the authors.

at the mathematical horizon refers to an awareness of how mathematical topics are related over the span of mathematics included in the curriculum.

This study examines pre-service teachers' content knowledge such as specialized content knowledge and knowledge at the mathematical horizon. That is, our study focuses on pre-service teachers' specialized content knowledge in fraction concept (part-whole construct) and fraction comparison, and how this specialized content knowledge of different topics are horizontally related to advanced fractional knowledge, which deals with multiplicative and reversible reasoning involving fractional quantities and composite units.

Method

Participants and Contexts

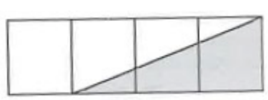
Data from this study came from 96 PSTs from two different university sites. Participants were either in their sophomore, junior or internship year of elementary teacher preparation programs—one from at a large northeastern university and the other from at a large southwestern in the United States. Each PST was enrolled in an elementary mathematics methods course. Mathematics methods courses were designed to support PSTs' knowledge development for teaching elementary mathematics.

Data Sources

A written task was used for the study, which asks three important fractional knowledge: (1) Fraction comparison (recognizing the difference between the actual numbers and fraction of the number and comparing the size of fractions); (2) fraction concept (the concept of referent units to represent a partial portion of a whole in given area models); (3) Multiplicative relationships involving fraction quantities and composite units (reversible reasoning including composite units of fractions). The provided questions are as follows (see Table 1).

Table 1: Main Task of this Study

1. At both Rivers High School and Mountainview High School, ninth graders either walk or ride the bus to school. 6/7 of the 9th grade students in Rivers High School ride the bus, while 7/8 of the 9th grade students in Mountainview High School ride the bus. If there are 40 9th grade students who walk at Rivers and 25 9th grade students who walk at Mountainview, in which school do more students ride the bus? In which school do a greater fraction of the students ride the bus? Explain your strategies or solutions as much as in detail.
2. For each picture shown below, (i) write a fraction to show what part is shaded. For each picture, (ii) describe in pictures or words how you found that fraction, and why you believe it is the answer.

(1)  (2) 

3. Merlyn spends $60 of her paycheck on clothes and then spends 1/3 of her remaining money on food. If she $90 left after she buys the food, what was the amount of her paycheck? Explain your solution method as much as in detail. You may use representations (e.g., diagrams, rectangles, number line etc.).

Data Collection and Data Analysis

Three questions about fractions were administered to the entire class in four mathematics methods course sections towards the end of the spring semester in 2014. Qualitative and quantitative analyses were conducted. In particular, for the written response, we used an inductive content analysis approach (Grbich, 2007). We initially organized raw data into an Excel spreadsheet, read all of the responses, and created codes based on the raw data. More specifically, data analysis involved five processes: (a) an

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Articles published in the Proceedings are copyrighted by the authors.

initial reading of each PST's response, (b) identifying correctness of the responses, (c) exploring the subcategories under each analytical aspect according to the number of correct responses and their problem solving strategies that PST demonstrated, (d) coding the categories and subcategories, and (e) interpreting the data quantitatively and qualitatively (Creswell, 1998).

Summary of Findings

In this section, we present overall findings from the written task and detailed analysis depending on PSTs' cognitive levels in fractional knowledge. Because we divided the cognitive levels according to the number of correct answers among three questions, it would be helpful to be aware of what fractional knowledge is involved in each three question. The first question is focused on comparing the size of fractions and recognizing the difference of comparing the size of the actual numbers and fractions. To find in which school a greater fraction of the students rides the bus, students need to compare the sizes of the given fractions 6/7 and 7/8 using procedural knowledge or conceptual knowledge. However, to find in which school more students ride the bus, students may use measurement construct by focusing on unit fraction and iterations (multiplicative relationship). That is, considering the given fractions for bus riders at two schools were 6/7 and 7/8, students would be able to know that 1/7 and 1/8 indicates fractions for walkers, which corresponds to the number of students given in the questions, 40 students at Rivers and 25 students at Mountain view. Thus, students can find the number of bus riders by multiplying 40 by 6 or 25 by 7 because 6/7 or 7/8 can be created by iterating a unit fraction, 1/7 or 1/8 six times or seven times, respectively.

The second question is focused on concept of fractions, more specifically, understanding of referent units to represent a partial portion of a whole in the given area models. To find a fraction of shaded parts in the diagram, students need to pay attention to what is considered as a referent unit and how to represent the different sizes of continuously or discontinuously shaded portions using the same referent unit (a whole). Depending on students' cognitive levels, some students could solve this problem by finding the total area or portion of the shaded part or by finding fractions of each shaded portion and adding them together, or by finding a fraction of the smallest portion, setting it as a unit and counting the total number of the unit through iterations.

The third question is focused on understanding of multiplicative relationships and reversible reasoning when composite units of fractions are included. To find the amount of original paycheck, students need to know 2/3 of the remaining money equals $90 because 1/3 of the remaining money was spent for food. Also, students need to understand that 2/3 is created by iterating a unit fraction (1/3) twice and thus the amount of money corresponding to 1/3 can be found by dividing $90 by 2. From here, students will be able to easily figure out the amount of money corresponding to 3/3 and so the original paycheck can be found by multiplying 45 by 3 and adding it to 60 via reversible reasoning.

Among the three questions, we think the third problem is most difficult followed by the first and second questions. The third question requires understanding fractions as operators and requires multi-steps to solve the problem. The first question may be solved by either using measurement sub-construct or ratio sub-construct. The second question, which may be easiest, can be solved by relying on part-whole sub-construct. Because each question that we asked was created by focusing on specific fractional knowledge, we first divided PSTs' cognitive levels depending on the number of correct answers and further examined their strategies used to solve the questions. More specifically, we divided PSTs into three categories depending on the number of correct answers. PSTs who got all three correct answers were sorted into level 3, and PSTs who got two questions were assigned to level 2. PSTs with all incorrect answers to the three questions were considered at level 0. 19 PSTs were assigned to level 3 and 36 PSTs were assigned to level 2. Also, there were 28 PSTs corresponding to level 1 and 16 PSTs corresponding to level 0. In the next section, we present each level in more detail.

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Articles published in the Proceedings are copyrighted by the authors.

PSTs at Level 3

PSTs at level 3 demonstrated following features in terms of problem solving strategies and fractional knowledge across the three questions (see Table 1). PSTs at level 3 demonstrated a good understanding of fractions as part-whole, measure, and operator in solving problems. In particular, they tended to often use multiplicative reasoning, which is related to iterations in the measure construct to solve word problems involving fraction quantities. Also, the PSTs showed that they had a clear concept of unknown and were able to create equations to find unknowns based on the concepts of unknown and the understanding of fractions as measure and operator.

Table 2: Features of PSTs at Level 3 in problem solving strategies and fractional knowledge

Level 3	Features
Fractional knowledge	• Understand the part-whole construct of fraction • Understand the measurement construct of fraction by using iteration (multiplicative reasoning) to solve the problems • Understand the operator construct of fraction by considering fractions as a multiplicative operator on knowns or unknowns • Have a concept of unknown and create equations to find the unknowns
Problem solving strategies	• Understand problem clearly • Often use drawing to make better sense of the problem situations • Tend to use multiple ways of solving the problems

Regarding fractional knowledge used to solve each question, in the first question, PSTs at level 3 compared the size of two fractions by using a conceptual strategy (i.e. 1/7 is bigger than 1/8 and thus 7/8 should be greater than 6/7 because 7/8 took out less fraction than 6/7), converting fractions into decimals (i.e. 7/8 [= 0.87] is greater than 6/7 [= 0.85]), or finding common denominators procedurally (i.e. 7/8 [=49/56] is greater than 6/7 [=48/56]). However, to find in which school has more bus riders, most PSTs tended to identify a unit fraction for walker from the given information and use the multiplicative relationship to figure out the number of bus riders at two schools (Figure 1).

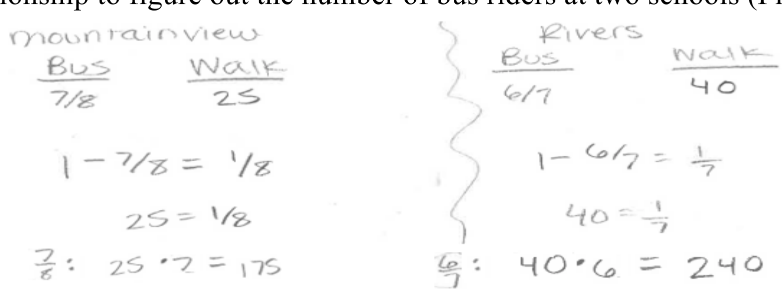
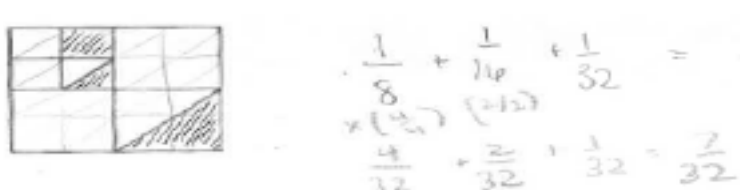


Fig. 1 Solution to find in which school has more bus riders in question 1.

Also, in solving the second question, most PSTs used a strategy to divide all parts into small equal sizes (1/16 or 1/32), set the equal size as a unit, and count the number of iterations of the unit (See Figure 2). This strategy shows that PSTs at level 3 understand measurement construct of fractions.



Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Articles published in the Proceedings are copyrighted by the authors.

Fig. 2 Solution to find a fraction of shaded parts.

Finally, in solving the third question, two-thirds of PSTs found the amount of original paycheck by using multiplicative relationship from measurement construct of fraction. For example, PSTs first found the amount of money corresponding to 1/3 of remaining money by using the multiplicative relationship that 2/3 can be created by iterating 1/3 twice and the given information that $90 is 2/3 of the remaining money. Then PSTs added the money spent on clothes ($60) to the entire remaining money, which consists of 2/3 ($90) and 1/3 ($45). Figure 4 presents PSTs' solution method where they use their understanding of unknown and operator construct of fractions. PSTs set variable *x* as the original paycheck that she wanted to find and then represented the amount of money spent on food as (x-60) × 1/3 by considering 1/3 of remaining money was spent on food after spending $60 on clothes.

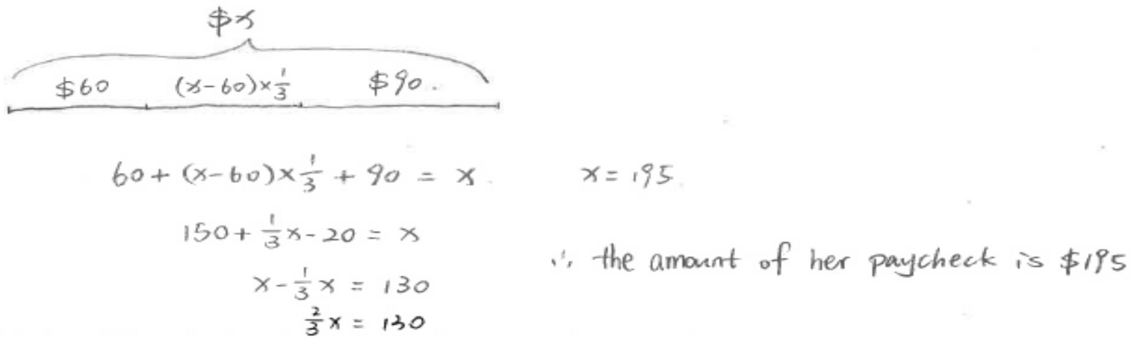


Fig. 3 Solution to use unknown and operator construct of fractions in question 3.

PSTs at Level 2

PSTs at Level 2 got two questions out of the three questions. We divided three sub-categories within level 2 depending on which questions PSTs got. For example, level 2-1 designates PSTs who got correct answers in both questions 3 and 2 (level 2-2: PSTs who got a correct answers in question 3 and 1 / level 2-3: correct answer in question 1 and 2). Although there are three sub-categories in level 2, we consider level 2-1 is highest, followed by level 2-2 and 2-1 due to the problem difficulty, which will be explained in detailed. We observed many similarities between PSTs at level 3 and those at level 2-1. 15 PSTs at level 2-1 demonstrate a good understanding of fractions a measure and an operator. However, these PSTs did not understand the problem correctly to question 1 and provide only one of the two sub-questions. With respect to problem solving strategies, they tended to represent the problem using drawing and solve each problem in multiple ways.

However, PSTs at level 2-2 seem not fully understand fractions as an operator. Although four PSTs provided a correct answer to the question 3, which requires multi-step multiplicative thinking, they often ignored denominators and solved it like whole number problems. 5 PSTs were categorized at level 2-1 where they were not able to recognize known and unknowns in correctly. In solving question 3, they showed difficulty in using measurement concept (e.g., 2/3 is twice of 1/3). In solving question #1, they relied on guess and check by either using given fractions.

PSTs at Level 1

PSTs at Level 1provide only one correct answer and consequently three subcategories exist in this level. For example, level 1-1 indicates PSTs who got a correct answer in question 1; level 1-2 includes PSTs who got a correct answer in question 2, and level 1-3 with PSTs who got a correct answer in question 3). Because each problem requires different level of problem difficulty, we consider PSTs at 1-3 is the highest level followed by those at level 1-1 and at 1-2. 13 PSTs were at 1-1; eight were at level 1-2; and six were at level 1-3. Table 3 shows the features of problem solving strategies and fraction knowledge at this level.

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Articles published in the Proceedings are copyrighted by the authors.

Table 3: Fractional knowledge and problem solving strategies by PSTs at Level 1

Level/Description	Features
1-1: Correct answer to question 1 (Middle)	*Fractional knowledge* • Be able to apply fraction as a ratio and measurement. • Yet, show lack of understanding of fraction as operator/ multiplicative thinking. *Problem solving strategies* • Although some used logical reasoning, rely on guess and check.
1-2: Correct answer to question 2 (Lowest)	*Fractional knowledge* • Develop an understanding of fraction as part whole. • Understand the measurement construct of fraction by using iteration (multiplicative reasoning) to solve the problems. • Lack understanding of fraction as measurement, operator, or ratio. *Problem solving strategies* • In solving # 2, relying on visualization by rearranging all parts together to form one simple fraction or viewing half of 3 squares. • Often do not understand the problem correctly. • Make computational errors/ Misrepresent the multiplicative relationship.
1-3: Correct answer to question 3 (Highest)	*Fractional knowledge* • Have some understanding of multiplicative thinking involving fractions. • However, have limited understanding of fraction as ratio or measurement. *Problem solving strategies* • Tend to use guess or backward/ Often do not recognize the problem correctly. • Misrepresent the second denominators.

PSTs at Level 0

15 PSTs were categorized at Level 0 where none of the answers are correct in the three fraction problems. In Question 1, these PSTs showed limited understanding of fraction as a ratio or measurement by either focusing on only one part of the two. In solving the question 2, they tended to miss the denominator and consider the problem as involving whole numbers. In solving Question 3, they showed lack of understanding of unknown and were not able to construct or solve algebraic equation. In particular, most of these pre-service teachers failed to identify the multiplicative relationship by recognizing the referent unit incorrectly.

Discussion and Concluding Remarks

In this study, we investigated how pre-service teachers solved three fraction problems that require different sub-constructs. By tracing their correctness and problem solving strategies in the three problems, we also explored how PSTs' use of fractional concepts is related to solving problems involving advanced fractional knowledge. First, we found that many of our PSTs developed limited understanding of fraction sub-constructs and thereby were not able to solve the three problems. Another finding is that when PSTs relied on only one sub-construct such as part-whole, they tend to provide incorrect answers. Based on our findings, we hypothesize that PSTs' understanding of fractions as measure and operator may be a foundation of solving problems involving advanced fractional knowledge.

References

Ball, D. L., Thames M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education 59*, 389-407.

Articles published in the Proceedings are copyrighted by the authors.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis—Emphasis on the operator construct. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13–47). Hillsdale, NJ: Lawrence Erlbaum.

Clarke, D. M., Roche, A., & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School,* 13(7), 373-380.

Creswell, J. W. (1998). *Qualitative inquiry and research design: Choosing among five traditions*. Thousand Oaks: Sage Publications.

Grbich, C (2007). *Qualitative Data Analysis: An Introduction*. London: SAGE Publications.

Johanning, D. J. (2008). Learning to use fractions: Examining middle school students' emerging fraction literacy. *Journal for Research in Mathematics Education, 39*(3), 281-310.

Ma, L. (1999).*Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*(3), 267–295. doi:10.2307/749828

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

Sibert, D., & Gaskin, N. (2006). Creating, naming, and justifying fractions. *Teaching Children Mathematics, 12*(8), 394-400.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *Journal of Mathematical Behavior, 22*, 237–295.

Streefland, L. (1991). *Fractions in realistic mathematics education*. Dordrecht, The Netherlands: Kluwer Academic

Usiskin, Z. (2007). Some thoughts about fractions. *Mathematics Teaching in the Middle School, 12*(7), 370-373.

Van de Walle, J. (2007). *Elementary and middle school mathematics: Teaching developmentally* (6th ed.). New York: Pearson Education.

Articles published in the Proceedings are copyrighted by the authors.