A CALL TO ACTION: TOWARDS AN ECOLOGICAL-DYNAMICS THEORY OF MATHEMATICS LEARNING, TEACHING, AND DESIGN

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Whereas Natural User Interface technological devices, such as tablets, are bringing physical interaction back into mathematics learning activities, existing educational theory is not geared to inform or interpret such learning. In particular, educational researchers investigating instructional interactions still need intellectual and methodological frameworks for conceptualizing, designing, facilitating, and analyzing how students' immersive hands-on dynamical experiences become formulated within semiotic registers typical of mathematical discourse. We present paradigmatic empirical examples of tutor–student behaviors in an embodied-interaction learning environment, the Mathematical Imagery Trainer for Proportion. Drawing on ecological dynamics—a blend of dynamical-systems theory and ecological psychology—we describe the emergence of mathematical concepts from the guided discovery of sensorimotor schemes.

Keywords: Cognition; Instructional Activities and Practices; Design Experiments; Technology

Introduction: In Search of Action-Oriented Theory of Mathematical Ontogenesis

Whereas commercial production of interactive math apps is booming, extant theory of learning is still a theory-of-learning-with-*paper*(Papert, 2004). In the short term, scarcity of bold research on interactive mathematics learning impedes the formulation of informed policies concerning the integration of technological environments into educational institutions. In the long term, this scarcity is accelerating misalignment between extant theory of learning and emerging practices to which it should apply. As children are learning to move in new ways, so, too, should theory of learning.

A motivation of this paper is that the pedagogical quality and institutional acceptance of action-based learning environments is largely pending on developing informed scholarly and public discourse concerning what it means to learn a mathematical concept and what an instructor's role might be in this process. As such, we are echoing Seymour Papert's consistent call to leverage the technological revolution as an opportunity for deep discussion of the potentially radical changes the educational system should undergo. Similar to Papert, we are optimistic that technological advances in educational media bear the potential of fostering students' deep understanding of mathematical concepts. Complementarily, we submit, these technological advances bear the potential of fostering researchers' deep understanding of learning processes.

A pedagogical rationale to ground mathematics learning in physical interaction echoes centuries of educational scholarship. We now sketch its recent history. From his cultural–historical psychology perspective, Vygotsky believed that meanings are established through physical interaction. Moreover, he asserted that mature mathematical reasoning tacitly retains and evokes its originary enactive quality (Vygotsky, 1926/1997, pp. 161-163). From a cognitive-developmental psychological perspective, Piaget (1971, p. 6), too, viewed thought as truncated action, emphasizing that "mathematics uses operations and transformations…. which are still actions although they are carried out mentally." Piaget (1968, p. 18)later introduced the notion of *action coordination* as the root of reasoning (see also Nemirovsky et al., 2013).From a philosophy perspective, a resonant view of thought as truncated action has been elaborated by Melser (2004), who puts forth aphylogenetic embodied model of language and reasoning. From an educational-research perspective, Skemp (1976) critiqued math instruction as fostering disjointed "instrumental" knowledge. He promoted an alternative educational program that instead would foster deep "relational" knowledge that resides in non-symbolical dynamical interactions. Similar, Pirie and Kieren (1994) advanced an Enactivist view of knowledge to implicate mathematical reasoning as drawing on dynamical imagery (see Reid, 2014). Decades later, Nathan (2012) denounced mainstream educational practice as still implicitly subscribing to a "formalisms first" epistemology and called to ground mathematical meaning instead in "our direct physical and perceptual experiences" (p. 139). Thompson (2013), too, points to the fundamental problem of mathematics education as the absence of *meaning*, that is, webs of multimodal imagery actions. These inspirational fiats leave us with a set of questions: How do naïve goal-oriented actions give rise to reasoning about immaterial entities? How do students first accept cultural signs? In particular, how might this transpire in discovery-based instruction?

Granted, a number of theoretical frameworks from the learning sciences have been formative in modeling artifact-mediated guided learning of STEM content, such as instrumental genesis (Vérillon & Rabardel, 1995), professional perception (Stevens & Hall, 1998), cultural anthropology (Hutchins, 2014), and semiotic approaches (Radford, 2014). However, these frameworks are not optimally geared to treat the new forms of pedagogical, technological, epistemological, and interactional opportunities created by NUI embodied-interaction learning environments. In particular, extant theoretical frameworks lack analytical specificity for treating sensorimotor schemes—how they emerge, how they are steered, and how they give rise to conceptual knowledge—as the phenomenal core of mathematics learning. And so we present a call to *action* as our *Critical Response to Enduring Challenges in Mathematics Education* (PME-NA 37).

We hasten to note up front that our focus in this paper on fostering motor-action coordinations should not for a moment suggest that we are disregarding or mitigating the formative role of symbols in the development of mathematical knowledge or disavowing the rich theoretical and practical challenges that the symbolic register introduces (Duval, 2006). Rather, we believe that there has not been sufficient focus in the literature on the initial development of action schemes via direct or vicarious interaction with instructional media (but see de Freitas & Sinclair, 2012). And we view NUI technologies as powerful yet under-researched means of fostering those action schemes. Accordingly, this article treats the initial guided construction of mathematically oriented operatory schemes more so than the subsequent signification of these schemes in disciplinary semiotic systems.

Empirical Context: Design-Based Research of the Mathematical Imagery Trainer

The *Kinemathics* project (Reinholz et al., 2011) took on the design problem of students' enduring challenges with proportional relations. We assumed that students have scarce sense of what proportional equivalence is, feels, or looks like. We began by choreographing a bimanual motor-action scheme that enacts proportional equivalence, and then we envisioned, designed, and engineered conditions in which students could learn to move in a new way that emulates this scheme. Our two-step activity plan was for students to: (1) develop a target motor-action scheme as a dynamical solution to a situated problem bearing no mathematical symbolism; and (2) describe these schemes mathematically, using semiotic means we then interpolate into the action problem space.

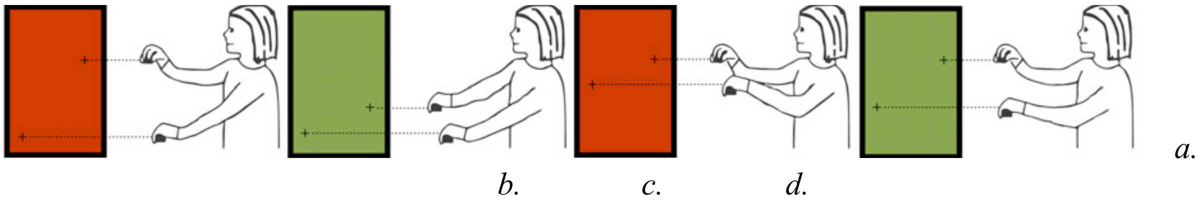
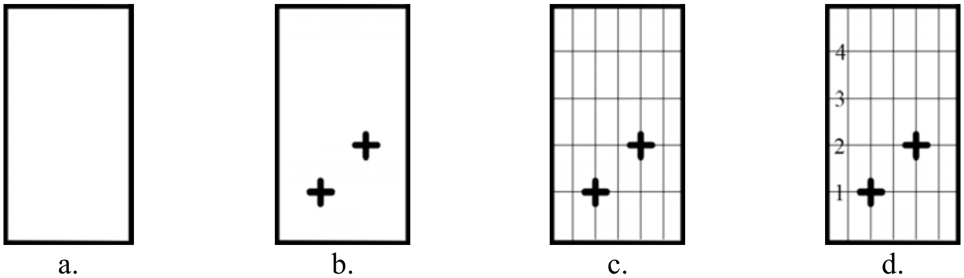


Figure 1. The Mathematical Imagery Trainer for Proportion (MIT-P)

Figure 1 shows the MIT-P set at a 1:2 ratio, so that the favorable sensory feedback (a green background) is activated only when the right hand is twice as high along the monitor as the left hand. This figure sketches out our Grade 4 – 6 study participants' paradigmatic interaction sequence toward discovering an effective operatory scheme: (a) while exploring, the student first positions the hands incorrectly (red feedback); (b) stumbles upon a correct position (green); (c) raises hands maintaining a fixed interval between them (red); and (d) corrects position (green). Compare 1b and 1d to note the different vertical intervals between the virtual objects.

Our design solution was the Mathematical Imagery Trainer for Proportion (MIT-P, Fig. 1). We seat a student at a desk in front of a large, red-colored screen and ask the student to "make the screen green." The screen will be green only if the cursors' heights along the screen relate by the correct ratio (e.g., 1:2). Participants are tasked first to make the screen green and then to maintain a green screen while they move their hands.

The activity advances along a sequence of stages, each launched when the instructor introduces a new display overlay immediately after the student has satisfied a protocol criterion (Fig. 2). The full design includes a ratio table for students to control the cursors indirectly via inserting numbers. (For an iPad version, seewww.tinyurl.com/FreeMITP).



a. b. c. d.

Figure 2. MIT-P display schematics, beginning with (a) a blank screen, and then featuring the virtual objects (symbolic artifacts) that the facilitator incrementally overlays onto the display: (b) cursors; (c) a grid; and (d) numerals along the *y*-axis of the grid. For the purposes of this figure, the schematics are simplified and not drawn to scale.

We implemented the MIT-P design in the form of a tutorial task-based clinical interview with 22 Grade 4 – 6 students, who participated either individually or in pairs, and these sessions were audio–video recorded for subsequent analysis (Reinholz et al., 2011). Our primary methodological approach is for the laboratory's researchers to engage in collaborative ethnographic micro-analysis of selected brief episodes from the entire data corpus (Siegler, 2006), where we focus on the study participants' range of physical actions and multimodal utterance around the available media (Ferrara, 2014). The process is iterative and in dialogue with the learning-sciences literature, leading to the progressive identification, labeling, and refinement of emergent categories (Strauss & Corbin, 1990). New constructs might constitute ontological innovations extending beyond the study context (diSessa & Cobb, 2004). Here we re-analyze our empirical data via a new lens.

Ecological Dynamics

Constructivist pedagogy champions the principle of fostering opportunities for individuals to re-invent cultural–historical knowledge (Kamii & DeClark, 1985). Yet how does this principle play out in learning environments where students are first to re-invent sensorimotor schemes prior to signifying the schemes in a discipline's semiotic register? We sought a theory of learning focused explicitly on the development of physical skill.

Ecological dynamics (Vilar et al., 2012) is a theoretical approach used in sports sciences to study skill acquisition in representative designs of real-game conditions. The framework blends *dynamical systems theory* (Thelen & Smith, 1994)and *ecological psychology* theory of affordances(Gibson, 1977).Applying dynamical systems theory to ecological psychology enables sports scientists to explain the learning of physical skills as the complex and adaptive self-organizing of subject–environment dynamical systems.

Dynamical systems theory is a branch of physics that provides a formal representation of any system evolving over time. The behavior of any living system can be plotted as a trajectory into a state space. In a dynamical systems approach, decision-making and learning processes are modeled not as generating a sequence of disembodied symbolical propositions, such as abstracted inferences and decisions, but as emerging from the agent's goal-oriented, situated, adaptive interactions in the environment (Araújo et al., 2009). The emergent quality of self-organizing complex adaptive systems implies also that learning processes are not linear but stochastic, and the non-linear dynamics of systemic behavior increases with the number of agents, variables, and interactions.

The self-organizing behavior of dynamical systems consisting of human agents (e.g., students) engaged in goal-oriented activity can be affected or "channeled" (Araújo & Davids, 2004, p. 50) by different types of constraints. Newell(1996) identified three sources of constraints affecting the behavior of the system either on a short time scale (i.e., decision making while performing a skill) or a longer time scale (i.e., the process of learning a skill): organism, environment, and task.

In terms of methodology, ecological dynamics may offer STEM educational research interpretive analytical schemes for modeling the role of instructors' multimodal utterance and actions in shaping students' construction of dynamical enactments. From its systemic view, ecological dynamics regards all forms of intervention, such as physical guidance or metaphoric framing, as productive constraints on the solution of motor-action problems.

The Ecological Dynamics of the Mathematical Imagery Trainer

We now present three ecological-dynamics accounts of children's guided work with an embodied-interaction design for mathematics content, the MIT-P. These concise narratives were selected as appropriate exemplars for showcasing numerous analyses of manipulation, discovery, and coaching in the context of math learning. For continuity, we will treat aspects of student behavior only around the numerical item of a 1:2 ratio.

The Emergence of an Attentional Anchor Mediating System Dynamics

Students typically begin the activity by lifting the controls and, in an attempt to make the screen green, waving them up and down in several different patterns. Eventually, the students discover that their hands "have to be a certain distance" from each other, and yet they attempt to keep this distance fixed. But as they further explore the screen regions, they figure out that "the higher you go, the bigger the distance" (Fig. 3a)Students thus discover, articulate, and empirically validate a systemic interaction principle governing a phenomenon under inquiry: a proposed correlation between two

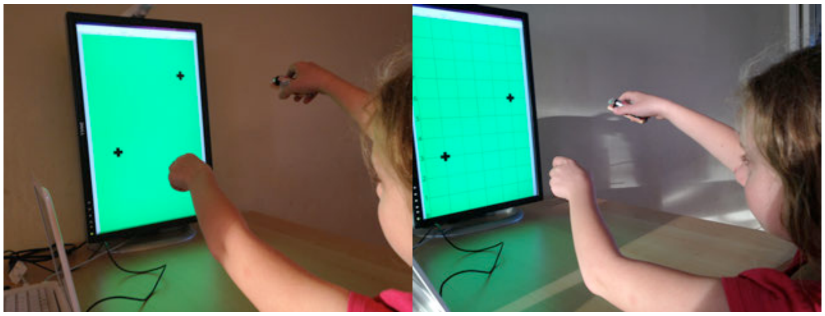

a. *b.*

Figure 3. (a) A child discovers the vertical interval between the markers as an attentional anchor for making green while moving the hands: the higher it is, the bigger it should be (and vice versa). (b) Once the grid is overlaid, she shifts spontaneously to a new routine.

qualitative properties of *a new object*—the height and size of a linear interval subtended between their hands.

We have been intrigued by students' initial discovery of the interval between their hands as a means of controlling the screen color as well as by the subsequent smooth shift from keeping this interval fixed as they elevate their hands along the screen to varying the interval size in proportion with its elevation. Crafted spontaneously out of thin air, the interval articulates into being, foregrounded from negative space as a new auxiliary stimulus wedged between agent and artifact. The interval coalesces as a ready-to-hand tool for engaging latent correlations in the perceptual field, thus mediating the situated implementation of motor intentionality. It served the students as a spontaneous self-constraint—an order parameter, "steering wheel" (Kelso & Engstrøm, 2006; Newell et al., 2010), or *attentional anchor* facilitating enactment (Hutto & Sánchez-García, 2014).

Decomposing and Recomposing an Attentional Anchor in a Reference Field

Once a student discovers the "the higher, the bigger" control strategy oriented on the interval between their hands as a new attentional anchor, the interview protocol proceeds to the next item, the introduction of the grid onto the screen (Fig. 3b). The appearance of the virtual horizontal gridlines materialized the imaginary attentional anchor. The grid's figural qualities immediately relieved students from having to hold the interval between their hands: The attentional anchor was thus electronically reified in the public domain in the form of a perceptually stable, externally present, deictically referable, bounded entity. Yet this frame of reference shifted students abruptly into a new interaction routine: raise the left hand 1 unit, then raise the right hand 2 units, iteratively, to make green. Now the old attentional anchor no longer mediated a goal, so it receded back into negative space.

The theory of ecological dynamics thus offers a view of conceptual development as spontaneous, situated adoption of symbolic artifacts *as action tools*. Symbolic artifacts bear hybrid ontology, in the sense that they are both perceptual and semiotic entities (Uttal, Scudder, & DeLoache, 1997). They are "transitional objects" (Papert, 1980)—both sensory and abstract. We might grab a symbol for its perceptuomotor affordance for action yet only subsequently—as personal and interpersonal situations evolve—leverage its semiotic potential for planning and communicating prospective actions, elaborating reasoning, and supporting argumentation. We kindle then obey new constraints.

Instructor's Multimodal Intervention as Environmental Constraints on Action

Learning is the education of perceptuomotor attention, and teachers can play pivotal roles in this educational process. One expert–novice co-enactment method is to distribute the operation of the

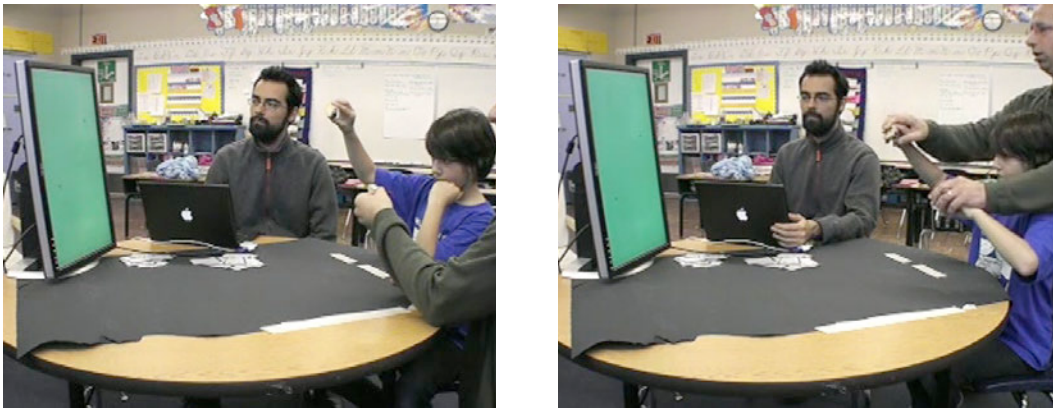

a. *b.*

Figure 4. Hands-on learning may need hands-on teaching:(a) co-manipulating virtual objects as distributed co-enactment; and (b) molding as joint co-enactment.

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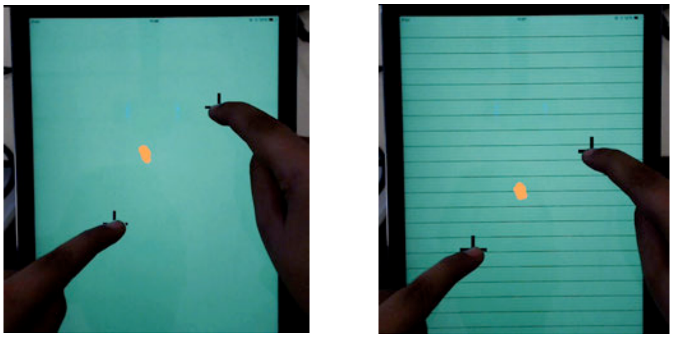
control devices, one person per device. Another method is to co-operate both of the control devices, that is, with both people each handling both devices.

Similar to a pair of athletes in a two-person sport, such as rowing or luge, the tutor and student optimize for effective joint production by continuously and dynamically adjusting each to the other's spatiotemporal actions (Fig. 4). They reach an intimate level of intersubjective sensorimotor coordination by anticipating and closely tracking their mutual actions. Yet as in the martial-arts practice of push-hands, these two participants silently negotiate leadership. The tutor progressively hands over agency to the student, who eventually solo-enacts the new strategy. The tutor-as-dynamical-scaffold fades out.



a. *b.*

Figure 5.Spontaneous evolution of an attentional anchor. The eye gaze (orange spot) hovers over a location on the screen that contains no information in-and-of-itself but only with respect to the dynamical motor-action coordination. A grid then offers a frame of reference for bringing forth the attentional anchor into mathematical consciousness.

Recent results from eye-tracking studies (Fig. 5) confirm our qualitative analyses of the interview data: Just before students articulate a new manipulation strategy, their visual attention tears away from figural constituents on the screen to anchor onto a new location, a higher-order invented "handle" on a structural constellation, that facilitates operable interaction with a dynamical yet invariant conservation (Shayan et al., 2015).

Conclusion

We have introduced an ecological-dynamics view on mathematics teaching and learning. We further presented and interpreted empirical data from implementations of embodied-interaction activities so as to contextualize the ecological-dynamics view and argue for its purchase on enduring research problems germane to the learning sciences in general and to scholarship and application of mathematics education in particular. Based on our findings, we contend that the theory of ecological dynamics offers a useful framework for designing, implementing, and analyzing pedagogical interactions in which students develop fundamental understandings of mathematical notions via solving and reflecting on motor-action inquiry problems. We also explained how an ecological-dynamics view of mathematics learning coheres with, integrates, and extends seminal constructivist and socio-cultural historical perspectives on human learning. From this view, mathematical meanings are cultural constructs that individual agents build by developing and then signifying appropriate motor-action coordinations oriented on discovered attentional anchors. These dynamical coordinations are embodied solutions to physical problems that students encounter when engaging in carefully designed activities.

We wish to position this article as attempting to rekindle essential themes of situated cognition (Greeno, 1998). To our reading, ecological dynamics should offer an effective and comprehensive framework for analyzing socially guided ontogenesis of intelligent participation in cultural practices. In particular, ecological dynamics, with its view of learning as coordinating motor actions in perceptual fields, may replace the cognitive semantics theory of conceptual metaphor (Lakoff & Núñez, 2000) as a more viable means of tracking the subjective and intersubjective emergence of mathematical concepts from situated solving of sensorimotor interaction problems (Gibbs, 2014). Ecological dynamics offers new tools for minding the epistemic gap between action and symbol and thus stands to fill a critical, enduring lacuna in mathematics-education research literature.

Given appropriate cultural mediation, children can learn quite rapidly to move and therefore think in new ways that become signified, elaborated, refined, and reformulated as disciplinary discourse. This thesis suggests *all* children's universal capacity to deeply understand mathematical concepts, regardless of prior academic accomplishment, because it shifts the site of critical mathematical learning away from the symbolic semiotic register toward situated sensorimotor engagement with manipulation problems.

Embodied-interaction activities offer solutions for researchers and teachers alike who wish both to observe mathematical thinking as it is occurring and offer students opportunities to reflect on their actions. Technology-enabled embodied-interaction learning environments transform the practice of mathematical teaching, rendering it similar to coaching in disciplines more readily associated with physical action, such as music, dance, or carpentry. Yet for these instructional devices and methodologies to enter educational institutions, we would all have to rethink multiple aspects of mathematics teachers' professional practice, beginning from epistemology and through to assessment.

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