THE NARRATIVE STRUCTURE OF MATHEMATICS LECTURES

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Although lecture is the traditional method of university mathematics instruction, there has been little empirical research that describes the structure of lectures. In this paper, we apply ideas from narrative analysis to an upper-level mathematics lecture. We develop a framework that enables us to conceptualize the lecture as consisting of collections of narratives, to identify connections between the narratives, and to use the narrative structure to identify key features of the lecture. By looking at repetitions of the mathematical concepts across the narratives, graph theory tools provide a means of examining the structure of the lecture. The analysis highlights the demands that students may face to understand the connections between the various mathematical ideas that the instructor introduces.

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Lecture is the traditional method of university mathematics instruction. Although there have been a few attempts to describe these lectures (e.g., Dreyfus, 1991), Speer, Smith, and Horvath (2010) noted that, due to the dearth of empirical studies, most of our beliefs about the structure of lectures are based on popular opinion or personal experience.

In general, we view mathematics lectures as consisting of instantiations of mathematical entities (i.e., objects, concepts, and symbols) in service of enacting and modeling broader mathematical processes (such as conjecturing, representing, and justifying) and habits of mind (e.g., looking for patterns). The goal of the current study is to create and use a framework based on narrative analysis to analyze a lecture, identify and describe the various processes and habits of mind that are modeled by the instructor, illustrate the ways the mathematical entities evolve and transform over the course of the lecture, and describe the ways in which the entities and processes are connected to each other.

Theoretical Framework

Narrative analysis, a common analytical technique in literary theory (e.g., Bal, 2009; Holley & Coylar, 2012), considers the sequencing of the elements of the text and how initial elements influence and shape later events (Holley & Coylar, 2012) and focuses on the evolving relationship between characters. Several researchers have described the close relationship between mathematics texts and narratives (e.g., Netz, 2005), and others have directly applied concepts from narrative analysis to analyze aspects of mathematical texts (e.g., de Freitas, 2012; Dietiker, 2012; Andrà, 2013).

Although there are numerous definitions of what a narrative is, they commonly attend to the temporal ordering of events, the way in which these events are connected, and the meaning that is ascribed to the sequence of events by a particular audience(see, e.g., Czarniawska, 2004; Reissman, 2005). We adopt the perspective of Holley and Colyar (2009), who identify narratives as texts in which a reader or observer can identify events, characters, and a plot—and sees connections between the events of the narrative.

Event structure

Bal (2009) describes the interrelated ideas of event, story, and fabula: The narrative text conveys events—"transition[s] from one state to another state" (p. 6); the sequence of events in the text make up the story; and the fabula is the "series of logically and chronologically related events that are caused or experienced by actors" (p. 5). Dietiker (2012) adapted these ideas to apply to mathematical texts:

- A *mathematical event* is "a transition from one mathematical state to another" (p. 15), such as instantiating a mathematical object, creating a representation, or making a conjecture.
- A *mathematical story* is "the [temporal] sequence of events encountered and experienced by a reader throughout a mathematics text" (p. 15).
- The *mathematical fabula* is "a reader's reorganization of the logic around how certain mathematical ideas support or connect the meaning of other mathematical ideas" (p. 16).

Dietiker (2013) thus conceptualizes a mathematical text as a narrative through the lens of comparing the mathematical story and fabula. While the former describes the chronological sequence in which an instructor presents mathematical concepts, the latter describes the logical relationships between the concepts.

Mathematical Characters

In the previous research on mathematical narratives, there has not been a consensus on what constitutes a character (e.g., Dietiker, 2013; Andrà, 2013; de Freitas, 2012). We describe *characters* in a mathematics lecture as including (but not limited to) the mathematical objects and concepts that the instructor instantiates that play a role in the story; these objects can be either general (e.g., the concept of an equivalence class) or specific (e.g., "the equivalence class of 3"). A character is identified by both the underlying concept and its representation (e.g., "the integer a" and "the integer b" are different characters).

Mathematical plot

The plot of a narrative consists of the meaningful connections between the events and informs the construction of a fabula (Polkinghorne, 1988). We define the *plot* of a mathematics narrative to be a description of the way a mathematical process or habit of mind is applied to a collection of mathematical objects or concepts. For example, in the lecture described here, the instructor used integers to illustrate a property of a specific equivalence relation, and then used this example to explain properties of equivalence relations in general. We can describe this process using the habit of mind "articulating a generalization using mathematical language" (Mark, Cuoco, Goldenberg & Sword, 2010).

Identifying Narratives in Mathematics Lectures

Boundaries of narratives. According to Labov (1972), a narrative includes, at a minimum, a sequence of two temporally ordered clauses. The process of identifying the "boundaries" of narratives—where the story begins and ends—can be nontrivial. To address this for the case of mathematics lectures, we identify the initiation and conclusion of a narrative by a shift in the presented content or mode of operation of the class, the use of board space, or natural language speech cues (e.g., the instructor saying "Now…" or "Let's consider…").

Identifying plot and characters. The plot and characters are not found "neatly packaged as such by the narrator" (Emden, 1998, p. 35). Rather, plots and characters are identified through a "tacking procedure" (Polkinghorne, 1988, p. 19) or a process of abduction (Czarniawska, 2004). This begins with the proposal of a potential plot and characters, and then these are compared with the events of the story to see how well they provide a coherent theme.

Framing and embedded narratives. In some cases, narratives are contained within other narratives and function as events within the containing narrative. This can be described by the ideas of *framing narratives* and *embedded narratives*, which describe a hierarchical relationship between two narratives. Following Ryan (1991) and Palmer (2004), we describe the role of a framing narrative as providing a context for its embedded narratives. The plot of the framing narrative provides coherence and focus for the plots of the embedded narratives; thus, the framing narrative influences the way we might read (or observe) an embedded narrative. Conversely, an embedded narrative typically serves as an event in a framing narrative.

Logical connections between events. Although a fabula must be consistent with the plot of a narrative, as Dietiker (2012) notes, "there are many deductive lines of reasoning that can lead to the same conclusion" (p. 16). From a broad perspective, we view two narratives or events as connected when one mathematically builds upon the other. Typical plots of a mathematical narrative include defining and redefining, constructing patterns, and forming and verifying conjectures; such plots may intuitively establish mathematical connections with other narratives by applying these mathematical processes to characters shared between two narratives. Thus, we attempt to capture the idea of mathematical building from one narrative to another by saying that narrative A *connects* to narrative B if a character in narrative B is first formally defined or instantiated in narrative A. Using this definition, the two narratives are connected if the plot of B involves mathematical processes that are applied to the same mathematical objects and concepts that are introduced in the plot of A. These logical connections form the basis for identifying a fabula for a narrative.

Methods

Data Collection

The lectures were taken from a standard junior-level abstract algebra course at a large university in the northeastern United States. The instructor was a tenured professor with a research focus in algebra; he had previously taught the course numerous times. We selected four lectures from the beginning, middle, and near the end of the semester and included instances of presenting definitions, examples, theorems and proofs.

A member of the research team attended each of the lectures and took notes on the instructor's speech and writing. Each lecture was video-recorded; we transcribed the instructor's speech verbatim from the recorded video and used the video to check the accuracy of the researchers' notes. For the analysis presented here, we selected a 45-minute excerpt from one of the lectures; this portion of the lecture included instances of definitions, examples, theorems, and proofs, and the instructor's presentation style was similar to the other lectures. Thus, we view this portion of the lecture as representative of the entire corpus of data.

Identifying Narratives and Plots

To identify characters in the lecture, we began by looking for instances where the instructor instantiated a mathematical object or concept, using the instructor's gestures, speech, and writing to inform this identification.

In order to identify potential framing narratives, we employed a "top-down" approach by identifying a coherent, meaningful plot and its constituent characters. We read the transcript of the lecture holistically and identified broad themes along with the mathematical objects, characters, and representations that were the primary focus of these themes. To identify potential embedded narratives, we employed a "bottom-up" approach by identifying sequential clauses that, when taken together, appeared to describe or apply a mathematical process to a set of characters.

To identify potential plots from these themes and characters, we looked for mathematical processes and habits of mind that the instructor employed, as well as definitions, theorems, and processes that typically play an important role in an abstract algebra class and might serve as a focal point for processes and habits of mind. Then, we worked abductively by identifying the characters and events and then revising the proposed plots so that they provided coherence to the chronologically related elements.

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Results and Analysis

Embedded Narratives

The primary focus of our analysis is the collection of framing narratives in the lecture. Consequently, we present here only a summary of the plots of the embedded narratives and other events of the framing narratives, as shown in Table 1.

Table 1: Embedded Narratives

Narrative Number	Description of Plot or Events
Framing Event 0	The instructor identifies the "goals" for the lecture and provides motivation for each goal: examining equivalence relations and introducing the idea of equivalence classes.
1	The instructor recalls and summarizes a narrative from a previous class: "R4" (i.e. for $a,b \in \mathbb{Z}$, $a\sim b$ if $\|a\|=\|b\|$) is an equivalence relation
2	The instructor starts to construct an alternative method for representing an equivalence relation and applies it to the R4 equivalence relation: The set of integers is represented by a large circle; each integer is represented by a labeled dot inside the circle; and a line is drawn between two dots when the corresponding integers are equivalent under ~.
3	The instructor poses the question: Can there be a triangle (i.e. three distinct dots that are all connected to each other) in the dot-and-line diagram?
4	The instructor poses and answers the question: Can there be line segments (i.e. three distinct dots where one dot is connected to the other two) in the dot-and-line diagram?
Framing Event1	The instructor describes R4 as relating pairs of numbers
Framing Event2	The instructor describes motivation for introducing the idea of an equivalence class.
5	The instructor elaborates the dot-and-line diagram by introducing "loops"
6	The instructor poses and answers the question: Does introducing "loops" violate the "no line segments" condition?
Framing Event3	The instructor describes additional motivation for introducing the idea of an equivalence class
7	The instructor generalizes from a "concrete" equivalence relation to an abstract one by translating the dot-and-line diagram for R4 into a dot-and-line diagram for an "abstract" equivalence relation

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Framing Event4	The instructor presents a general, formal definition of equivalence class
8	The instructor generalizes from the previous examples to create a precise mathematical description of equivalence classes for R4
9	The instructor challenges his previous work and works to make more precise his definition of equivalence classes. He poses and answers the question: Are equivalence classes well defined?
10	The instructor poses and answers the question: Are there any equivalence classes under R4 with a single element?
11	The instructor conjectures that [3]=[-3] under R4 (this example is subsequently generalized to all integers). He then tests this conjecture, setting up a proof by contradiction by asking whether [3]≠[-3].
12	The instructor conjectures and proves that [a]=[-a] for any integer a under the equivalence relation R4
13	The instructor conjectures and then sets up a proof that, for integers a and b, if a~b then [a]=[b] under the equivalence relation R4

Framing Narratives

At the beginning of the class, the instructor described the "goals" he had for the day and wrote them on the board: Identifying general properties of equivalence relations, introducing the idea of an equivalence class, and identifying general properties of equivalence classes. Based on this, as well as a holistic reading of the transcript, we identified the three framing narratives as shown in Table 2.

Table 2: Framing Narratives

Framing Narrative	Plot
A	Creating and refining an alternative representation: The instructor develops a diagram for representing a particular equivalence relation
B	The instructor generalizes from a "concrete" example to develop the concept of an equivalence class and construct a precise, abstract definition.
C	The instructor conjectures and then proves that two equivalence classes that share an element are equal

Connections Between Narratives

Figure 1 shows the connections between the various embedded narratives and other events in the lecture; the shape of each vertex indicates the framing narrative of which it was a part, as shown in

Figure 2. In the graph drawing, each vertex corresponds to an embedded narrative or a non-narrative event, labeled with the corresponding number from Table 1. The edges between the vertices

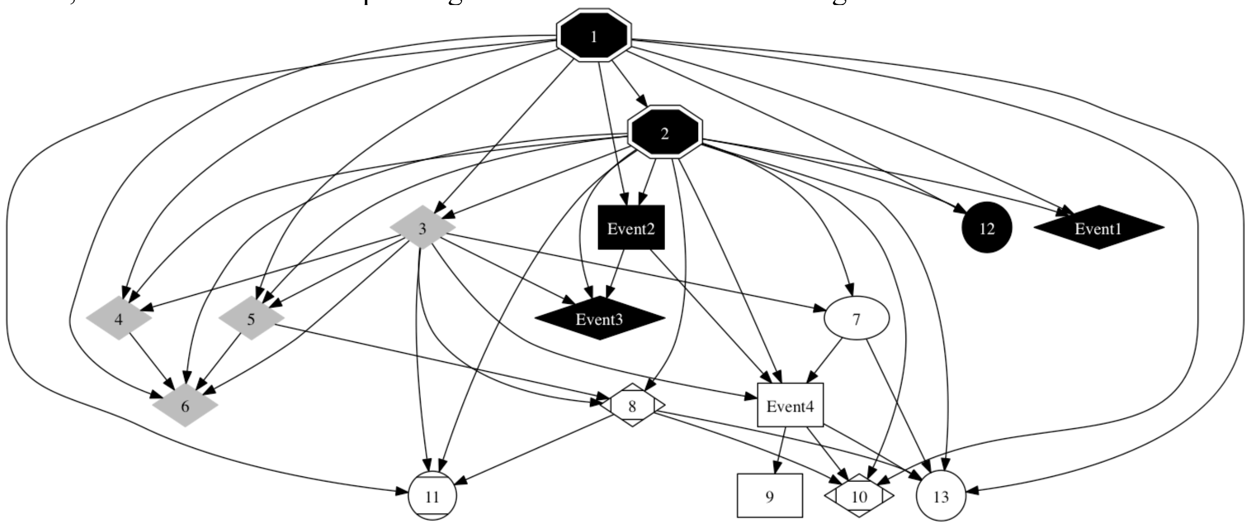

Figure 1: Diagram of Connections Between Embedded Narratives and Events

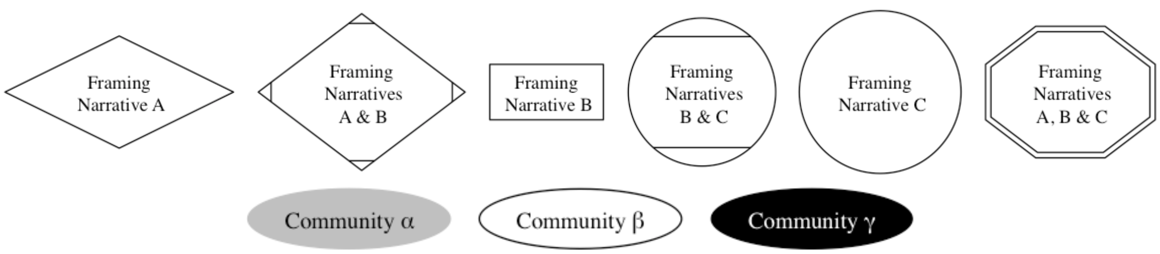

Figure 2: Legend For the Diagram of Connections

represent the connections, and the connection X ➔ Y indicates Y includes characters that were introduced in X.

Structure of the Graph

There are numerous methods for analyzing the graph to identify structural aspects of the lecture. One such method is to identify the *community structure* of the graph; this partitions the graph so that the collection of narratives in each subgraph is relatively densely connected but the connections between the subgraphs are relatively sparse. Using a modularity maximization algorithm, the graph in Figure 1 can be split into the three communities, which are indicated by the shading of the vertices as indicated in Figure 2. There is a close alignment between the communities and framing narratives, which suggests that each framing narrative has relatively dense internal connections and, consequently, relatively strong internal coherence

The graph also displays how "closely related" various pairs of embedded narratives are—i.e., how directly the characters introduced in one narrative are included in a subsequent narrative. This can by identified by analyzing the various paths between vertices. For example, fully understanding the plot of EN10 indirectly required an understanding the characters in embedded narratives 8, 5, 4, and 3 and the relationship between these characters. This "chaining" is reflected in the length of the directed path between embedded narratives 3 and 10.

Another aspect of the lecture highlighted by the graph-theoretic structure is the "centrality" of various embedded narratives, which reveals their narrative significance. There are numerous types of graph-theoretic centrality measures that can be used. For example, the *pagerank centrality* of a vertex measures the number of connections from other vertices and gives a higher weight to connections from other highly connected vertices (Brin & Page, 1998). A narrative with high pagerank centrality has numerous direct and indirect connections to previous narratives; it integrates characters from numerous other narratives and is likely to be culmination of many paths through the graph of embedded narratives. Embedded narratives 6 and 13 had the largest page rank centrality. Both of these narratives integrated the characters from numerous other narratives: EN6 combined R4, the dot-and-line diagram, and the idea of follow-on lines to refine the diagram; EN13 drew upon most of the mathematical objects in the lecture to prove a conjecture about the equality of equivalence classes.

Discussion

In contrast to the common notion of lectures as consisting of formal sequences of definitions, theorems, proofs, and applications (Dreyfus, 1991), the results described here suggest that mathematics lectures may have a significantly more complex structure.

We hypothesize that thinking about parts of a lecture in terms of plots and characters is useful for researchers to assist in identifying the main mathematical ideas of the lecture and the different ways that those ideas may be connected and built upon by the various mathematical objects and concepts. Identifying elements of embedded and framing narratives—including the characters, plots, complications, and resolutions—enables us to describe the temporal development of the important mathematical ideas in the lecture.

In addition to being useful for researchers, identifying the plots and connections between the narratives can help us better understand the challenges that might arise in learning from a lecture. In particular, the complexity of the narrative structure and the numerous habits of mind that are used in the plots suggest reasons why students—who might not possess these habits of mind or be able to quickly identify the structure—might struggle to make sense of a lecture.

Constructing meaning from a text by viewing it as a narrative is a different process than employing a logico-scientific mode of knowing. From a narrative perspective, events are given meaning through abduction; the only way to identify the plot or structural elements of a narrative is by "negotiating and renegotiating meanings by the mediation of narrative interpretation" (Bruner, 1990, p. 67). Consequently apprehending the plot of a framing narrative can involve a complex process in which students must identify the multiple roles that can be played by each embedded narrative, identify potential orderings of these roles, and understand how they fit together.

The analysis of the connections between the embedded narratives revealed several important aspects of the lecture. First, the community structure analysis showed that each framing narrative contained relatively dense connections. If these connections are not made explicit, it may be challenging for students to identify the ways in which the embedded narratives are related to each other. Conversely, these connections may also support sense-making: If students can understand aspects of one narrative, then they can use its connections to develop an understanding of the connected narratives or to identify the plot of a framing narrative. In addition to this structure analysis, the centrality analysis can reveal the ways narratives synthesize ideas, serve as key focal points, and incorporate characters that play significant roles in the narratives.

As long as lecture remains a common form of mathematics instruction, more research is needed to describe lectures themselves as well as the ways students interpret and learn from lectures. The data reported here come from a case study and may not necessarily be representative of all mathematics classes—or even all abstract algebra classes. In order to more completely describe mathematics lectures, it will be essential to gather a larger corpus of data—both more classes from a single instructor, and lectures from other instructors. It will also be essential to collect data on students' interpretations of lectures and to determine what students learn from each lecture and how the narrative structure—and their interpretation of the structure—relates to this learning.

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