LINGUISTIC NORMS OF UNDERGRADUATE MATHEMATICS PROOF WRITING AS DISCUSSED BY MATHEMATICIANS AND UNDERSTOOD BY STUDENTS

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We studied the linguistic norms of mathematical proof writing at the undergraduate level by asking two mathematicians and five mathematics undergraduate students to read seven partial proofs based on student-generated work and to identify and discuss uses of mathematical language that were out of the ordinary with respect to standard mathematical proof writing. By asking participants to discuss the seriousness of each breach, we not only identify and discuss some of these linguistic norms, but also describe important differences between the ways in which mathematicians and students understand them.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking

In professional mathematical practice, proofs are an essential type of communication. In an influential paper on the role of proof in mathematics, Rav (1999) wrote that proofs "are the heart of mathematics" and that they play an "intricate role […] in generating mathematical knowledge and understanding" (p.6). As a result, fostering undergraduate mathematics students' abilities to understand and construct valid proofs is one of the primary goals of mathematics instruction at the advanced undergraduate level. However, evidence of undergraduate students' difficulties when reading and constructing proofs is pervasive in the mathematics education literature (Weber, 2003). One potential difficulty students have when constructing proofs concerns students' inability to understand and use mathematical language and notation (Moore, 1994).

Mathematical language has been studied and interpreted in a variety of different ways: as a foreign language composed almost entirely of technical symbolic representations (Ervynck, 1992), as a combination of natural language and a system of mathematical symbols (Kane, 1968), and as a set of meaning that is created and expanded with the creation of terminology and the designation of technical definitions to natural English words (Pimm, 1987). In particular, researchers have focused on the differences between mathematical language and neutral or common language. For example, Veel (1999) discussed the precision necessary when implementing certain verb phrases in mathematics and Halliday (1978) noted the nominalization of mathematical language, in which a mathematical action or phenomenon becomes an object (e.g., differentiation). As these aspects of precision and rigor in mathematical writing may cause difficulties for students, a number of mathematics educators have suggested ways to improve students' use of mathematical language (e.g. Veel, 1999; Moschkovich, 1999; Lemke, 2003). However, these suggestions have focused on school level mathematics. Research on how mathematicians and undergraduate students understand the language of mathematics is lacking: to our knowledge, there are only two studies (Konior, 1993; Burton & Morgan, 2000) to date that have explicitly and empirically investigated the language of mathematical proof writing, and neither study investigates how undergraduate students understand such language.

Objectives of the Study

This qualitative study is a first attempt to address this gap in the literature by examining how mathematicians and undergraduate students understand the linguistic norms of mathematical proof writing. By interviewing both mathematicians and undergraduate students, this study not only identified various linguistic norms of undergraduate mathematics proof writing, but also illustrated how students understand these norms. In particular, this study addressed the following research

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questions:

1. How do mathematicians view and describe the linguistic norms of mathematical proof writing at the undergraduate level?
2. How do undergraduate mathematics students taking an introduction to proof course understand these norms?

Theoretical Framework

This study is informed by Herbst and Chazan's (2003) body of work on practical rationality. In particular, Herbst (2010) described norms as statements that articulate practice, as made by an observer of the practice. Since participants may not be fully aware of the norms they follow, Herbst and Chazan (2003) adapted the ethnomethodological concept of breaching experiments (Mehan & Wood, 1975) to study these norms. Herbst and Chazan hypothesized that when a participant of a practice is engaged in a situation where a norm has been breached, the participant will attempt to repair the breached norm highlighting not only what the norm is, but also expounding on the role that the norm has in the practice (Herbst, 2010). Adapting this methodology, this study used the concept of breached norms to investigate the linguistic norms of mathematical proof writing at the undergraduate level.

This study also employed the use of Scarcella's (2003) conceptual framework for academic English. Scarcella defined academic English as a "register of English used in professional books and characterized by the specific linguistic features associated with academic disciplines" (Scarcella, 2003, p. 9). This framework has previously been applied to mathematics education with regards to English learners studying mathematics (e.g. Silva et al., 2008; Heller, in press). However, we propose to apply the framework as a tool to begin to investigate important aspects of the mathematics sub-register of academic English. The framework of academic English specifies that there are three dimensions of language; the linguistic dimension, the cognitive dimension, and the sociocultural-psychological dimension. For the sake of brevity, we focus on the linguistic dimension and its components.

The linguistic dimension of academic English involves phonological, lexical, grammatical, sociolinguistic, and discourse components. The phonological component "includ[es] stress, intonation, and sound patterns" (p. 11) and "knowledge of graphemes (symbols) and arbitrary sound-symbol correspondences" (p.13).

The lexical component requires knowledge of the words used in a field. In particular, Scarcella (2003) distinguished between general words used in everyday language, academic words used across academic fields, and technical words that are field-dependent. She also included knowledge of fixed expressions, which are "expressions that tend to stick together and cannot be changed in any way" (p. 14), as part of the lexical component.

The grammatical component of academic English entails knowledge of "the grammatical co-occurrence relations that govern the use of nouns" (Scarcella, 2003, p.15). For instance, Scarcella (2003) noted students need to learn the associated grammatical features for these technical words, that "certain nouns […] are generally followed by prepositional phrases" and that some "verb + preposition combinations […] cannot be changed" (p. 16).

The sociolinguistic component involves developing competence in a variety of functions of language, including an understanding of the appropriateness of a given sentence in a particular context. Scarcella (2003) noted that "signaling cause and effect, hypothesizing, generalizing, comparing, contrasting, explaining, describing, defining, justifying, giving examples, sequencing, and evaluating" (p. 18) are examples of different academic language functions.

The discursive component entails understanding and using linguistic forms necessary to communicate successfully and coherently. For instance, in every day language, greetings and parting phrases indicate to speakers the beginning and end of conversations. Scarcella (2003) noted academic English “includes specific introductory features and other organizational signals” and that “writers’ presentation of ideas must be orderly and convey a sense of direction” (p. 19).

This conceptual framework of academic English highlights important aspects of learning an academic language, and the components of the linguistic dimension provide a lens for analyzing mathematicians’ and undergraduate mathematics students’ understandings of the linguistic norms of proof writing in undergraduate mathematics.

Methods

Based on the methodology used by Herbst (2010) and Herbst and Chazan (2003), this study investigates the linguistic norms of undergraduate mathematics proof writing by showing participants student-generated proofs and asking them to identify and describe uses of mathematical language that are out of the ordinary with respect to standard mathematical proof writing at the undergraduate level. By identifying these non-standard uses of mathematical language, mathematicians and undergraduate students discuss their understanding of the linguistic norms of proof writing at this level.

Two mathematicians and five undergraduate students were interviewed for this study. Both mathematicians had experience teaching an introduction to proof course at the large research university in the United States. The undergraduate students were all enrolled in an introduction to proof course at the same university at the time of the study. Interviews with individual participants were conducted by the first author, videotaped, and lasted one to two hours.

Materials

The materials for this study include seven partial proofs that are based on student-generated work and truncated to help participants focus on the use of mathematical language and not the attempted proof’s logical validity. One example of a partial proof used in the study is presented in Figure 1.

Let R and S be relations on a set A. Prove: $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

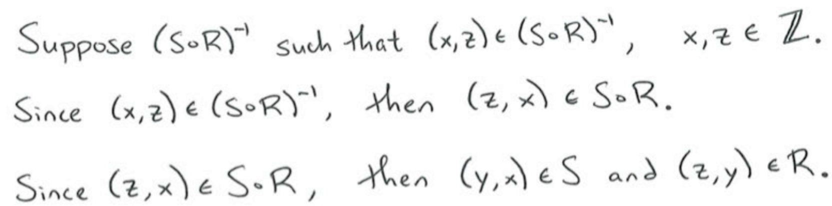

Figure 1: Example of the partial proofs presented to participants

These partial proofs were chosen from student exams given in an introduction to proof course at the same university of the study. For each one of these proofs we created a copy with markings of what we considered to be breaches of linguistic norms of mathematical proof writing at the undergraduate level. An eighth partial proof was constructed to illustrate the interview procedure prior to the beginning of the interview.

Procedures

Mathematicians were presented with the student-constructed partial proofs one at a time and were asked to mark the partial proofs for anything that was out of the ordinary with respect to the use of mathematical language in the writing of mathematicians.

The interviews made two passes through the materials. In the first pass, mathematicians were

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asked to explain why they made each mark indicating an unconventional use of mathematical language. For each of those unconventional uses, the mathematician was asked if the issue at hand was a logical issue, if it would affect the validity of the proof, if it was an issue of mathematical writing, if it was definitely unconventional or a matter of personal preference, if it significantly lowered the quality of the proof, and if the mathematician would have deducted points based on this issue when grading the student generated proof in an introductory proof course. These prompts were designed to elicit the mathematician's views on mathematical language with respect to proof writing. In particular, the prompts addressed the severity of each breach and enabled a differentiation between issues of logic and issues of mathematical writing in the analysis of the data.

In the second pass through the data, mathematicians were presented with marked copies of partial proofs presented in the first pass. That is, they were presented with a copy of a partial proof marked with one instance of an unconventional use of mathematical language as identified by the authors. These marked partial proofs were presented for each of the predicted instances of unconventional use of mathematical language that had not been previously identified by the participant in the first pass of the data. In particular, mathematicians were asked whether or not they would agree with a colleague of theirs who had suggested that these were indeed unconventional uses of mathematical language. If the participant agreed, the interviewer would then prompt the mathematician to discuss the breach as in the first pass of the data.

The structure of the interviews with the undergraduate students mirrored the structure of the interviews with mathematicians. However, the instructions and prompts used in the student interviews specified that the participants were to indicate and describe what they believed a mathematician would find out of the ordinary with respect to the use of mathematical language in the writing of mathematicians in formal settings. The only two other differences with the previously described protocol was that in the first pass students were not asked to make an assessment regarding the deduction of points in the grading of the partial proofs, and that in the second pass they were asked whether or not they would agree with a *classmate* of theirs who had suggested each unconventional use of mathematical language not identified in pass 1.

Analysis

Interview videos were transcribed and materials generated in the interviews were scanned for analysis. Data was analyzed using memoing and grounded theory in the style of Strauss and Corbin (1990). Memos were used to conceptualize the breached linguistic norms and to categorize the broken norms.

Results

The analysis of the mathematicians' interviews identified ten breaches of linguistic norms in the partial proofs presented to them. The number of mathematicians and the number of students who identified each one of these breached norms in the first part of the interview (pass 1) are listed in Table 1. Analysis of the interviews provided us with participants' descriptions of each breached norm, their perceived seriousness of each breach (whether they considered the breach to be definitely unconventional, or a matter of personal interest or context), and in the case of mathematicians, whether or not they would deduct points when grading a proof containing such a breach. A summary of some of this information is included in Table 1. However, for the sake of brevity, we only describe how four of these ten norms emerged from the data and how the student participants understood these four norms.

Include Necessary Antecedents

In the interviews, both mathematicians identified an unclear referent in a partial proof. Specifically, the proof included the word ‘it’ referring to a function that had not been explicitly defined in the partial proof. One of the mathematicians said:

Table 1: Identification of Norms and Their Severity by Mathematicians and Students in the First Pass

Linguistic Norms of Mathematical Proof Writing	Mathematicians*	Students**
Specify each variable	2 – 2 – 2	2 – 1
Include necessary antecedents	2 – 2 – 2	0 – 0
Use proper imperative sentence structure	2 – 2 – 2	4 – 3
Use standard mathematical vernacular	2 – 2 – 2	3 – 2
Write in full sentences	2 – 2 – 2	0 – 0
Make relations between statements clear	2 – 2 – 1	3 – 3
Do not use abbreviations in formal settings	2 – 2 – 0	4 – 1
Do not use formal propositional logic	1 – 0 – 0	0 – 0
Explicitly indicate the structure of a proof	1 – 1 – 1	1 – 1
Use notation appropriately in texts	1 – 1 – 0	3 – 1

*Number of mathematicians (out of 2) who identified the norm breach in the first pass (1st number), deemed the breach definitely unconventional (2nd number), and would deduct points for this type of breach (3rd number).
**Number of students (out of 5) who identified the breach of the norm in the first pass (1st number) and deemed the breach definitely unconventional (2nd number).

> Yes, I mean it’s always, in writing in general, you need to have… the pronouns have to have an antecedent. And so, not just mathematical writing, but in particular you have to be careful about mathematical writing.

In this quote the mathematician indicated the necessity of including antecedents to avoid unclear referents and suggested that it is not only the case that the rules of English grammar apply to mathematical language, but also that the adherence to this rule in mathematics is particularly important. Using Scarcella’s (2003) framework, we identify the inclusion of necessary antecedents as a linguistic norm regarding the grammatical component of the language of mathematical proof writing. Both mathematicians judged the use of pronouns lacking clear referents as definitely unconventional in mathematical proof writing and indicated that they would deduct points from an exam in an introductory proof course based on that breach.

However, none of the students identified the unclear referent in the first pass through the data. After presented with the marked partial proof indicating the unconventional use of mathematical language, four students agreed that the word ‘it’ introduced ambiguity. But these students did not see this issue as severely as the mathematicians. In particular, the students indicated that they believed a mathematician would view this issue as a matter of personal preference. Failing to view the grammatical necessity of including antecedents could be interpreted as evidence that these students are not proficient in this aspect of the grammatical component of mathematical proof writing.

Use Proper Imperative Sentence Structure

The interviewed mathematicians indicated that the following phrases from the partial proofs were ungrammatical and meaningless: 1) “Suppose $(R \circ S)^{-1}$ s.t. $(x, z) \in (R \circ S)^{-1}$” and 2) “Let $\forall n \in \mathbb{Z}$”. These are both imperative phrases beginning with transitive verbs. As such, English grammar dictates that the phrases need both a direct object and an object complement to be a

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complete sentence. That is to say, that the sentence must suppose the direct object in relation to another object or a property about the direct object.

Both mathematicians judged these two statements to be definitely unconventional and worthy of deducting points on an exam in an introduction to proof course. In particular, one mathematician said that the former "just doesn't make any sense at all, I don't know what they were… what they mean to be saying." He continued to say "Suppose what about $(R \circ S)^{-1}$? Following it with the 'such that' is not a statement about $(R \circ S)^{-1}$, so you can't say suppose", indicating that the sentence is incomplete. Concerning the phrase "Let $\forall n \in \mathbb{Z}$", the second mathematician said:

> It's not a statement. I mean, let something implies that the something is a statement that's being assumed and this is not a statement. […] Linguists probably have a word for this; it's just not a sentence. […] It's not a correctly constructed meaningful thing.

Here the mathematician attempted to describe that using the word 'let' needs to be followed by a meaningful statement. While the mathematicians had difficulty explaining the breached linguistic norm exactly, it is evident that both believed the phrases included unconventional use of mathematical language.

Four of the student participants agreed in the first pass that these sentences are unconventional uses of mathematical language. Moreover, one of the students indicated that the statements were incomplete sentences and were grammatically incorrect. However, they did not agree with the mathematicians' severe opinions of this type of breach. While three students indicated that this was indeed an unconventional use of mathematical language (and not simply a matter of writing style), each of students also believed such a use of language was harmless to the quality of the proof.

Make Relations Between Statements Clear

Both mathematicians discussed the importance of using verbal connectives to show the relations between different statements within the partial proofs. Moreover, they both agreed that lacking these verbal connectives was definitely unconventional and would merit point deductions on an exam in an introductory proof course. In particular, one mathematician said:

> No, it's not harmless. […] I mean if I'm worried about a student actually getting broken of a habit of making these kinds of, here's a statement, here's a statement, here's a statement, without drawing the connectives in between. Then that's not a logical issue, but it's a serious presentation issue.

In this quote, the mathematician indicated that he did not want students to write proofs as a series of unrelated statements, and deemed this breach as a serious presentation issue. Such relations between statements show the flow of the argument to readers, which is an aspect of the discursive component of mathematical language.

However, there was less agreement among the students. The three students who indicated in the first pass that one must make relations between statements clear each agreed that this use of mathematical language was definitely unconventional. One student said:

> I don't know if he's just stating it or if he means it to be part of that [assumption], but I think if you're writing it out formally, you should be more clear about it, like put an assume.

This student indicated that lacking verbal connectives could lead to ambiguity, which puts unnecessary stress on the reader to decipher the flow of the argument. On the contrary, two of the five students indicated in the second pass that there was no need for verbal connectives and a proof lacking words was conventional. One of these students said, "in some homework assignments, I have

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done this before and it's not wrong. You could write it in words or you could write it like this." This student did not seem to believe in the necessity of connecting the various statements in a proof.

Use Notation Appropriately within Text

One partial proof included the sentence "So there are 19 possible differences in $\mathbb{N}\times\mathbb{N}$ that are ≥ -9 and ≤ 9". One of the mathematicians indicated in the first pass that this sentence used the mathematical symbols $\geq$ and $\leq$ inappropriately. In particular, the mathematician said:

I would say that those should have been said in words rather than using the symbols. Or complete expression – that is, there are 19 possible differences, *d*, such that $d \geq -9$ and $d \leq 9$. […] Do it in a symbolic phrase or do it in words, but don't kind of mix it in.

In this quote, the mathematician indicated that the binary operators $\geq$ and $\leq$ require notation on both the left and right sides of the symbols. Relating the to lexical component of academic English, this quote suggests that using the binary operator without notation on both sides of the operator is an unconventional use of mathematical language. This mathematician indicated that inappropriately mixing the symbolic notation with text would significantly lower the quality of the presentation and that he would deduct points from an exam based on this use of language. While the other mathematician agreed in the second pass that such a use would be unconventional, he believed that mixing the symbolic notation with text is a personal preference and would not deduct points off of an exam. This mathematician however discussed that one should not technically use this type of "short hand" in formal settings, but that it is commonly used despite it being unconventional. This indicates the sociological component of mathematical proof writing includes understanding the appropriateness mixing mathematical notation and prose.

Three of the five students' responses in the first pass also indicated that mixing the notation and text in this way was an unconventional use of mathematical language. For example, one student said:

> I think it's just more the notation. It looks odd to just have the greater than sign without having something like directly in front of it. So I think it should be written out in words.

In this quote, the student described that the binary operators $\geq$ and $\leq$ require notation on both the left and right sides of the symbols, suggesting that the students agreed with the first mathematician on what was the appropriate way to use symbols in prose. On the other hand, many of the students also believed that this mix of notation and prose was harmless. In particular, one student said that "it's faster than writing greater than or equal to or less than or equal to [… and] I've seen lots of teachers use that when like they do lectures." So this student emphasizes that since teachers mixed the notation and prose when writing on the board, there is no reason why he should not do so as well.

Discussion

As this qualitative study considers only a small sample of mathematicians and undergraduate students, the findings are simply suggestive of how mathematicians and undergraduates view the language of undergraduate mathematics proof writing. In particular, based on mathematicians' interviews we have identified ten linguistic norms of mathematical proof writing. Our analysis suggests that while interviewed students showed some competence in certain aspects of the lexical and grammatical components of mathematical proof writing, there were significant differences between these students' understandings of some aspects of the sociological and discursive components of mathematical proof writing and the corresponding understandings of the mathematicians interviewed in this study.

While the nature of this study precludes any claim of sample-to-population generalization of these findings, we believe the linguistic norms of mathematical proof writing identified in this study, as well as the method suggested for studying such norms, opens interesting avenues for future

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research in an area that is both under-researched in mathematics education and important in our attempt to understand students' difficulties reading and writing mathematics. In particular, these findings suggest the following research questions: To what extent does the larger community of undergraduate mathematics professors agree on the linguistic norms of undergraduate mathematics proof writing identified in this study? To what extent do students in transition to proof courses (and beyond) agree on the described views of these norms, and to what extent do students' perceived views of these norms align with those of the professional mathematical community? How do students' understandings of these norms develop and change throughout a semester of an introduction to proof course? We are currently designing studies that address some of these questions.

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Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.