GENERALIZING AVERAGE RATE OF CHANGE FROM SINGLE- TO MULTIVARIABLE FUNCTIONS

Allison Dorko
Oregon State University
dorkoa@onid.oregonstate.edu

This paper explores students' ways of thinking about the average rate of change of a multivariable function and how they generalize those ways of thinking from rate of change of single-variable functions. I found that while students thought about the average rate of change of a multivariable function as the change in the independent quantity with respect to the changes in the dependent quantities, they had difficulty determining a process to assign a value to that rate of change. Most tried to represent the average rate of change as a singular expression, generalizing the Δy/Δx expression to create expressions of the form Δz/[Δx and Δy], yet did not appear to have a sense of what they believed they were measuring. This suggests that quantitative reasoning, or lack thereof, was at the heart of the students' generalizations .A pedagogical implication of this research is that students' natural tendency to try to determine a singular expression for the average rate of change of a multivariable function could serve as useful as motivating the need to hold a variable fixed.

Keywords: Advanced Mathematical Thinking; Cognition; Post-Secondary Education

Introduction

While it is clear to experts how multivariable calculus topics are mostly natural extensions of single-variable calculus topics, how students come to see the relationship between ideas like function and rate of change in single- and multivariable calculus is not well understood. Though some recent advances have been made with regard to student thinking about these ideas, these studies are only preliminary (Kabael, 2011; Martinez-Planell & Trigueros, 2013; Trigueros & Martinez-Planell, 2010; Yerushalmy, 1997). Multivariable functions are used extensively in the sciences, engineering, statistics, and higher mathematics. It is imperative that we learn more about student understanding of multivariable functions, as they serve as the foundational tools by which students make sense of and represent relationships between quantities in complicated systems in these fields. I focus in particular on average rate of change in three dimensions for two reasons. First, a coherent image of it is necessary to understand the mathematical construction of instantaneous rate of change. While average rate of change (e.g. average speed) exists in the real world, instantaneous rate of change is a mathematical construction that relies on anticipating the result of taking averages rates of change over infinitesimally small intervals. Second, since multivariable calculus topics like rate of change build on single-variable calculus topics, I think it is important to study students' understanding of the multivariable topic with respect to their understanding of its single-variable counterpart. That is, I focus on how students generalize ideas, which yields insight into both what students understand about a new idea and how they use prior knowledge in the process of making sense of new content. To address these two aims, I sought to answer the following research question: *How do students think about the average rate of change of a multivariable function and how is this generalized from their understanding of the average rate of change of a single-variable function?*

Background Literature

Findings from literature indicate many difficulties that students have understanding rates of change. Thinking about rate of change as a measurement of how fast quantities are changing is foundational to calculus, yet many students have difficulty reasoning about rate in this way (Rasmussen, 2001; Thompson & Silverman, 2008). Thompson (1994) proposed that understanding constant rate of change depends on coordinated understandings of respective accumulations of

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

accruals in relation to total accumulations. The understanding of constant rate of change he described entails quantities having covaried (Saldanha & Thompson, 1998; Thompson, 2011). This is not at all obvious to students. Rather, some students interpret the average rate of change of a single-variable function as the arithmetic mean of some number of instantaneous rates of change (Bezuidenhout, 1998; Dorko & Weber, 2013). Students may also struggle with computing $\Delta y/\Delta x$ to estimate the rate of change of linear and non-linear functions (Orton, 1983). Thompson (1994) and Zandieh (2000) have suggested that these difficulties may be attributed to not conceiving of rate of change as a ratio, but instead thinking of it as the steepness of a function.

To the author's knowledge, there are two studies about students' thinking about slopes and rate of change in three dimensions. McGee and Moore-Russo (2014) found that at the beginning of instruction about slope in three dimensions, students asked to determine the slope between the points (1,2,1) and (3,2,5) computed $m = \Delta y/\Delta x$. Some of these students found it difficult to understand that this formula did not work in three dimensions. Once students accepted that $m = \Delta y/\Delta x$ did not work in three dimensions, they agreed that the "rise" of the slope should be Δz, but were not sure what the "run" was. Weber (2015) also investigated how students conceived of rate of change in three dimensions, and found that students often sought a way to combine the rates of change of two independent quantities into a single rate of change. My work extends these studies' findings by focusing on students' generalizations of rate of change in addition to their conceptions of it.

Theoretical Perspective

My theoretical perspective drew from two sources, both with constructivist underpinnings. First, I used Ellis' (2007) framework for studying generalization. She defines generalization as the influence of prior activity on novel activity, even if the student's action in the new activity is not mathematically correct. It is important to note that by framing my work in terms of Ellis' framework, I implicitly adopt an actor-oriented perspective (Lobato, 2003) for characterizing students' ways of thinking, which sets aside normative notions of correctness and allows us to focus on *how* students make sense of a situation rather than the *outcome* of that sense-making. Ellis' framework characterizes students' generalizing activity in terms of *generalizing actions* and *reflection generalizations*, which we explain later in the methods. Second, I drew from Thompson's (1990) work on quantitative reasoning, which characterizes a specific way of conceiving of situations. Two key ideas from his work that I draw on here are quantity (a measurable attribute of an object), and quantification (the process of assigning a value to that attribute), which help to explain the different meanings for average rate of change that students generalized, and the different ways in which they attempted to calculate average rate of change.

Data Collection and Analysis

I conducted hour-long semi-structured interviews with eleven students currently enrolled in integral calculus at a large university in the Pacific Northwest. While eleven students is not a particularly large sample size, I believe that these results have what Maxwell (1996) calls *face generalizability*, or there is "no obvious reason *not* to believe that the results apply more generally" (p. 97, emphasis original). That is, even a case study of a few students is likely to generate results that apply more generally. Interviews were recorded with LiveScribe technology, which gives a synched record of students' written work and talk. Interviews were subsequently transcribed for use in coding. In this paper, I focus on students' responses to the following two tasks:

[Q1] Let $V(s) = s^3$ represent the volume of a cube. What is the average rate of change of the volume of the cube if the length of its sides increases by one?

[Q2] Let $A(L,W) = LW$ represent the area of a rectangle. What is the average rate of change of the area of the rectangle if the length increases by one and the width increases by two?

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

I chose area and volume as contexts for talking about average rate of change because I hypothesized that they would be novel contexts and hence allow me to observe students in the process of generalizing. If students became stuck, I asked them one or both of the following problems, which we thought would be more familiar:

[QA] What is the average rate of change of $f(x) = x^2$ over the interval [0,5]?

[QB] Suppose you are in a car and you travel from mile 324 to mile 360 in one hour. What is your average speed?

The $V(s)$ and $f(x)$ function tasks (and other similar area and volume tasks not reported on here) allowed me to observe how students thought about the average rate of change of a single-variable function, while the $A(L,W) = LW$ task (and other similar area and volume tasks not reported on here) allowed me to observe how students thought about the average rate of change of a multivariable function. Comparing students' responses from the single-variable to the multivariable tasks allowed me to analyze how students generalized their thinking about average rate of change from one setting to the other.

I used Ellis' (2007) generalization taxonomy as an analytic framework. This taxonomy distinguishes between *generalizing actions*, or students' mental activity as they generalize as inferred through their activity and talk, and *reflection generalizations,* or students' final statements of generalization. In this paper, I focus only on students' generalizing actions (Table 1) because these reveal student thought during the process of understanding a situation, while reflection generalizations are often summary statements of a generalization. Analysis consisted first of reading the transcripts and coding instances of generalization. I then re-read those instances, and coded them based on the categories shown in Table 1. Due to space limitation, I give examples for only the categories that appeared in this study. The examples in Table 1 are from Ellis' (2007b) paper.

Table 1. Ellis' Generalization Taxonomy (adapted from Ellis, 2007a; 2007b) (Ellis, 2007)

Generalizing Actions		
Type I: Relating	1. Relating situations: The formation of an association between two or more problems or situations.	Connecting back: The formation of a connection between a current situation and a previously-encountered situation. (Example: Realizing that "this gear problem is just like the swimming laps problem we did in class!")
		Creating new: The invention of a new situation viewed as similar to an existing situation.
	2. Relating objects: The formation of an association between two or more present objects.	Property: The association of objects by focusing on a property similar to both. (Example: Noticing that two equations in different forms both show a multiplicative relationship between x and y).
		Form: The association of objects by focusing on their similar form. (Example: Noticing that "those two equations both have one thing divided by another")
Type II: Searching	1. Searching for the same relationship: The performance of a repeated action in order to detect a stable relationship between two or more objects.	
	2. Searching for the same procedure: The repeated performance of a procedure in order to test whether it remains valid for all cases.	
	3. Searching for the same pattern: The repeated action to check whether a detected pattern remains stable across all cases.	
	4. Searching for the same solution or result: The performance of a repeated action in order to determine if the outcome of the action is identical every time.	

Type III: Extending	1. Expanding the range of applicability: The application of a phenomenon to a larger range of cases than that from which it originated. (Example: Having found that the difference between successive *y*-values is constant for $y = mx$ equations, applying the same rule to $y = mx + b$ equations)
	2. Removing particulars: The removal of some contextual details in order to develop a global case.
	3. Operating: The act of operating upon an object in order to generate new cases.
	4. Continuing: The act of repeating an existing pattern in order to generate new cases.

Results

I found that students primarily thought about the average rate of change as a ratio of changes (the measurement process or quantification), which measured some aspect of an object (either the graph or the growing rectangle). The students made sense of the average rate of a multivariable function by connecting back to prior situations; expanding the range of applicability of average rate of change as the ratio of the change in the independent quantity with respect to the change in the dependent quantity; and relating objects based on their form and property. For example, V6 tried to create an expression to represent her understanding of the average rate of change as meaning "how much is the area growing in regards to the changes in the length and the width". V6 struggled with what to put in the denominator of the expression, and concluded that 3 (the sum of ΔL and ΔW) made sense (Figure 1). The crossed-out part in V6's work is $[(L+1) - L] + [(W+2) - W]$, which indicates that she sought to make an expression of the form ΔA/[an expression she believed combined ΔL and ΔW]. After telling the interviewer that having two different variables was confusing her, V6 crossed out her first expression and wrote 3 because "the total change you're adding one to the length and two to the width and so the total change is three". It was unclear whether she recognized that her original denominator simplified to 3, or if she were not paying attention to that expression and instead thinking about a way to combine ΔL and ΔW.

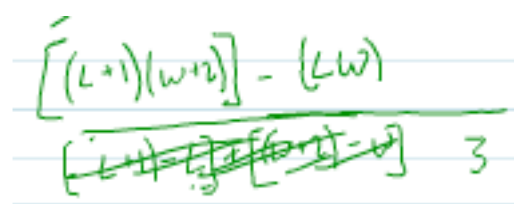

Figure 1: V6's Average Rate of Change of *A*(L,W)

V10 also constructed an expression of the form ΔA/[an expression she believed combined ΔL and ΔW], choosing to put coordinate pairs in the denominator (Figure 2). She began by writing what she believed to be the expression for the average rate of change of $f(x)$ [see top left of Figure 3] and then tried to construct a version for the average rate of change of $f(L,W)$ [top right of Figure 3], saying "I guess it would just translate over into two variables like that." This statement indicates that she related average rate of change in 2D and average rate of change in 3D as similar situations, and the arrow between the expressions indicates generalization based on the expressions' form (relating objects: form; see Table 1). The lower half of Figure 3 shows V10's attempt to determine an average rate of change for some actual length and width measures (it was unclear why she switched from multiplying the length and width in the crossed-out [(9)(5)] – [(8)(4)] to adding the dimensions in [(9 + 5)] – [(8 + 4)]). It is also unclear why V10 wrote $f'(x_1)$, $f'(x_0)$, $f'(L_1,W_1)$, and $f'(L_0,W_0)$, but used $f(L_1,W_1)$ and $f(L_0,W_0)$ in her computation. Other students also thought that average rate of change

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

involved derivative values rather than function values. One student (V11) explained, "I envision derivatives when anything involving change occurs."

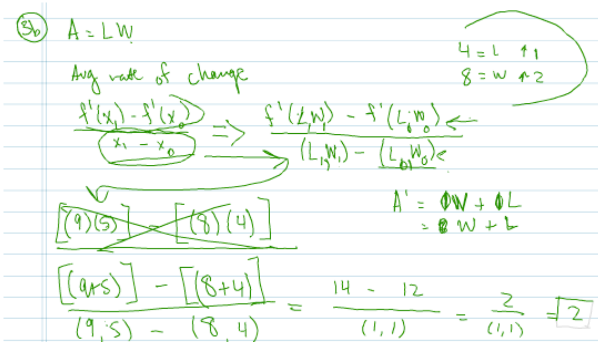

Figure 2: V10's Average Rate of Change for the Area of the Rectangle

While most students identified the average rate of change as meaning the change in area with respect to the change in the length and the width, V10 was the only student who came close to explaining that average rate of change is a *constant* rate. She said

> *V10*: A rate of change would be like at, like at any point it could be a different rate of change, I guess, and then the average rate of change would be more just like the mean rate of change. So like even if they, there might be some different, like some difference, some small difference between every point, you could take the average to try to predict a certain amount to add every time.

I interpret V10's comment about finding 'a certain amount to add every time' as evidence of thinking of average rate of change as a constant rate.

Discussion

Recall that I sought to answer the following question: How do students think about the average rate of change of a multivariable function and how is this generalized from their understanding of the average rate of change of a single-variable function? I found that students think of the average rate of change of a multivariable function as involving changes in three different quantities, and that they attempt to find a single expression to represent it. They generalize the form Δy/Δx to Δz/[a combination of Δx and Δy]. Weber (2015) also found that students' first inclination for the rate of change of a multivariable function is to combine the rates of change of the independent variables.

There is not sufficient evidence to claim that the students think about the average rate of change of a function (multivariable or not) or as a quantification of how variables vary. Indeed, they primarily focused on the structural aspects of calculating the rate of change. Very few students hinted at any notion of many variables varying simultaneously, or that the average rate of change is a constant rate of change. This finding further suggests that rate of change, average or instantaneous, is not about the variation in quantities for students, which led to some of the surprising approaches they used to "find" the average rate of change. In many cases, the students did not have a robust enough image average rate of change as a measure of covarying quantities to support the important conceptual issues one encounters in measuring rate of change in three dimensions.

My finding that students attempted to find the average rate of change of *A*(L,W) by creating a three-space version of $\Delta y/\Delta x$ is similar to McGee and Russo's (2014) finding that students initially try to find the slope between (x_1, y_1, z_1) and (x_2, y_2, z_2) by computing $m = \Delta y/\Delta x$. A difference is that McGee and Russo's students ignored the z coordinate entirely, while our students created expressions

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

of the form Δz/[some combination of Δx and Δy]. In both cases, however, students tried to leverage their understanding of the 3D scenario by connecting back to their knowledge of slope in 2D. They paid particular attention to the structure of slope as a change in one variable divided by the change in a second variable, and tried to create an expression analogous to $\Delta y/\Delta x$. While this study only reports on the behavior of eleven students, that my finding is similar to what McGee and Moore-Russo (2014) observe supports that the way the students in this study thought and acted with respect to multivariable functions may be representative of students in general.

Students' search for a single expression was largely generalizing by *relating objects* (see Table 1). Generalization by focusing on objects seems to play an important role in students' thinking not only in this setting, but in other cases in which students try to generalize an idea from a familiar f(x) context to the unfamiliar f(x,y) context (e.g., Dorko & Weber, 2014). I hypothesize two reasons for students' attempts to create a single expression. One is that before multivariable calculus most of the concepts students deal with can be represented by a single expression or formula (piecewise defined functions being one exception). I hypothesize that students thus come to expect that there exists one expression for everything. A second reason is that students may not have considered the implications of being in three-space, namely that having two independent variables often necessitates holding one variable constant so that one can talk about the change in the other variable. Supporting this hypothesis, Kabael (2011) found that students' schema for R^3 is critical for their construction of multivariable functions. I think that students' experience in trying to determine what went in the denominator of their expressions could be pedagogically useful as motivating the need to hold a variable fixed. That is, instructors could begin instruction about rates of change of multivariable functions by giving students tasks such as the ones used in this study, letting students discover that it is difficult to talk about two changes at once, and then introduce the idea of talking about a rate of change in a direction. An additional pedagogical implication for precalculus courses and lower-division mathematics is to find other situations in which a single expression is inadequate. This might prevent students from coming to believe that there exists a single equation that describes any given situation.

Regardless of students' difficulty determining what to put in the denominator, it is notable that students tried to compute such an expression in the first place. That is, thinking of the average rate of change as the change in the function values over the changes in the independent variables indicates that multivariable calculus students conceive of average rate of change as a ratio, an understanding that is not always present in single-variable calculus students (Orton, 1983; Thompson, 1994; Zandieh, 2000). In particular, students in this study did not talk about rates of change as "steepness". That students in this study largely generalized their understanding of average rate of change based on the notion of rate-as-ratio reinforces the importance of students developing a robust understanding of slope as a ratio in algebra.

Acknowledgments

Research reported in this article was supported by NSF Grant No. DRL-1419973. Any recommendations or conclusions stated here are the authors' and do not necessarily reflect official positions of the NSF.

References

Bezuidenhout, J. (1998). First year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology, 29*(3), 389-399.

Dorko, A. & Weber, E. (2014). Generalising calculus ideas from two dimensions to three: how multivariable calculus students think about domain and range. *Research in Mathematics Education*.

Dorko, A. & Weber, E. (2013). Students' concept images of average rate of change. In Martinez, M. & Castro Superfine, A. (Eds.). *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Chicago, IL: University of Illinois at Chicago.

Ellis, A. (2007a). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *The Journal of the Learning Sciences, 16*(2), 221-262.
Ellis, A. (2007b). Connecting between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education, 38*(3), 194-229.
Lobato, J. (2003). How design experiments can inform a rethinking of transfer and vice versa. *Educational Researcher, 32*(1), 17-20.
Kabael, T.U. (2011). Generalizing single variable functions to two-variable functions, function machine and APOS.*Educational Sciences: Theory and Practice, 11*(1), 16.
Martinez-Planell, Rafael, & Trigueros, M. . (2013). Graphs of functions of two variables: results from the design ofinstruction. *International Journal of Mahtematical Education in Science and Technology, 44*, 663-672.
Maxwell, J. A. (1996). *Qualitative Research Design* (Vol. 41). Thousand Oaks, CA: SAGE.
McGee, D. & Moore-Russo, D. (2014). Impact of explicit presentation of slopes in three dimensions on students' understanding of derivatives in multivariable calculus. *International Journal of Science and Mathematics Education.* doi: 10.1007/s10763-014-9542-0
Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, *14*(3), 235-250.
Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*(1), 55-87.
Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebraic. Center for Research in Mathematics & Science Education: San Diego State University.
Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. In *Learning Mathematics* (pp. 125-170). Springer Netherlands.
Thompson, P.W., & Silverman, J. (2008). The concept of accumulation in calculus. In M.P. Carlson & C.
Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp.117-131). Washington, DC: Mathematical Association of America.
Trigueros, M., & Martinez-Planell, Rafael. (2010). Geometrical representations in the learning of two-variablefunctions. *Educational Studies in Mathematics, 73*, 3-19.
Weber, E. (2015). The two-change problem and calculus students' thinking about direction and path. *The Journal ofMathematical Behavior, 37*, 83-93.
Weber, E., & Dorko, A. (2014). Students' and experts' schemes for rate of change and its representations. *The Journal of Mathematical Behavior*, *34*, 14-32.
Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education, 28*(4), 431-466.
Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, *8*, 103-122.