STUDENT CONCEPTIONS OF WHAT IT MEANS TO BASE A PROOF ON AN INFORMAL ARGUMENT

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In this paper we explore how students construe what it means for an informal argument to be the basis of a formal proof and what students pay attention to when assessing whether a proof is based on an informal argument. The data point to some undergraduate mathematics students having underdeveloped conceptions of what it means for a proof to be based on an argument. These underdeveloped conceptions limit what students pay attention to during informal-to-formal comparison tasks and may have adverse effects on students' ability to use their own informal arguments to construct proofs.

Keywords: Post-Secondary Education; Reasoning and Proof

Introduction

The constructs we use throughout this paper revolve around the observation that proofs are expected to be written in a verbal-symbolic representation system but may not be generated wholly within that system. Following Weber and Alcock (2009) we refer to reasoning that stays solely within this system as syntactic reasoning, and reasoning that falls outside of it as semantic reasoning. Similarly, we conceptualize a formal proof as a deductive argument that establishes the result to be proven and conforms to the norms of the representation system of proof. This is a characterization of the end product, not the reasoning that led to it. Additionally, we conceptualize an informal argument as a deductive argument that establishes the result to be proven, but does not conform to the norms of the representation system of proof. We refer to the use of informal arguments to inform the construction of formal proofs, or more generally the process of using semantic reasoning to inform syntactic reasoning, as formalization.

Research relevant to formalization can be partitioned into two non-disjoint categories. The first category focuses on the role of semantic reasoning in informing proof productions. The second category examines the semantic-to-syntactic formalization process needed to use semantic reasoning to generate a formal proof.

Research falling into the first category has illustrated the important role that semantic reasoning can play in proof generation. Various types of semantic reasoning have been shown to inform proof generation (Gibson, 1998, Sandefur, Mason, Stylianides, & Watson, 2013, Zazkis, Weber, & Mejia-Ramos, in press). This first body of work has perpetuated the recommendation that students should use semantic reasoning during proof construction. This recommendation has gained considerable traction in mathematics education (e.g., Garuti, Boero, & Lamut, 1998; Raman, 2003).

A second set of studies has focused on the formalization process itself. This research has provided evidence that mathematics majors struggle to use semantic reasoning to inform proof generation (Selden & Selden, 1995; Alcock & Weber, 2010; Zazkis et. al., in press).

These two sets of studies point to a discrepancy between researcher recommendations (that students should generate proofs using informal arguments) and students' behavior and abilities. In order to better understand this discrepancy we examine what mathematics majors pay attention to when attempting to determine if a formal proof is based on an informal argument and how these determinations compare to normatively correct interpretations.

In order to operationalize these notions we consider a formal proof to be *based on* an informal argument if there is a mapping between these two chains of inferences that has two properties: (1) it

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is meaning preserving to the extent allowed for by the rules of the verbal-symbolic system, and (2) corresponding inferences (or chains of inferences) appear in the same order. We use the acronym FBI-judgment, to refer to student judgments of whether Formal proofs are Based on Informal arguments.

The Study

We are interested in what it means for a proof to be based on an informal argument from a mathematics major's perspective. Thus we create a model of what mathematics majors pay attention to when making FBI-judgments and use this model to create a plausible explanation for why these students may have difficulty with formalizing informal arguments. In particular, we want to illustrate how prioritizing a particular subset of attributes when making informal to formal comparisons influences how students view what it means for a proof to be based on an argument and by extension, affects their ability to formalize.

Participants

Participants were pursuing undergraduate degrees in mathematics at a large state university in the Northeastern United States and had completed a proof based second course in linear algebra, an introduction to proof course, and an introductory analysis course. Thus, the eight participants were familiar with reading and writing proofs in a variety of mathematical contexts. Participants were selected to have roughly equal amounts of A, B, and C grades represented.

Procedure

The second author conducted one-on-one clinical interviews with each participant that lasted between 90 and120 minutes. This involved presenting participants with both informal arguments and proofs and engaged them in a series of comparison tasks. More specifically, he presented participants with triples that consisted of one informal argument and two formal correct proofs, only one of which, from our perspective, was based on the argument.

The informal argument in each triple was presented in the form of a video. Each video lasted approximately 30 seconds and involved the first author justifying the result with a combination of verbal argumentation, graph generation and gestures. The two correct formal proofs in each triple were presented in written form.

At the beginning of each one-on-one interview participants were told that they would be shown triples. They were told that they were to understand and compare the three parts of the triples, but that they did not need to validate correctness. In the first round of the interview participants were shown each of the three triples one at a time. They watched the video and read the two proofs out loud. After participants verified that they felt they understood the result, they were asked to make side-by-side comparisons of what they noticed in terms of similarities and differences between the three parts of the triple. This was done with each of the three triples. Note that during the first stage only the participants' impressions and what they noticed was elicited.

The second stage involved revisiting each triple and judging whether each of the proofs in the triple was based on its informal argument, how confident they were in their assessment and on what information they based their conclusion. Students were not informed that, from the researchers point of view only one of the proofs in each triple was a formalization of the informal argument. This made it possible for participants to conclude that neither or both of the proofs in each triple were based on the informal argument.

Analysis

A set of minimum criteria, which was agreed upon prior to the interviews, was used to determine whether student FBI-judgments were consistent with mathematical norms. We were also interested in

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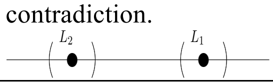
what students paid attention to when making FBI-judgments. Each interview was transcribed and grounded theory (Strauss & Corbin, 1990) was used to categorize interviewee responses in terms of what they paid attention to during FBI-judgments.

Materials

We briefly mention that Task 1 involved proofs that $\int_{-a}^{a} \sin^3(x)\,dx = 0$, for all real numbers a, and Task 3 involved proofs that the derivative of a differentiable even function is odd. For space reasons we discuss only Task 2 in detail. To clarify our discussion each step in the informal arguments is labeled with an "I2," each step in the formalization of this argument is labeled with an "F2," and each step in the distractor proof is labeled with a "D2."

<table>
<tr><th colspan="2">If a sequence (a_n) has a limit, it is unique.</th></tr>
<tr><th>Informal argument</th><th>Proof F_2 (Formalization of informal argument)</th></tr>
<tr><td rowspan="2">(I_2-1) If the limit wasn't unique then we could have two limits, say L_1 and L_2.
(I_2-2) But we control the size of the ε-neighborhood around these.
(I_2-3) and if we make it small enough these two ε-neighborhoods are going to not overlap.
(I_2-4) So when we get far enough down the sequence we're going to be both in here [pointing an neighbor hood around L_1] and in here [pointing an neighbor hood around L_2].
(I_3-5) But since we can't be in two places at the once we get a contradiction.

</td><td></td></tr>
<tr><td>Proof D_2 (Distractor)
(F_2-1) Toward a contradiction assume the sequence (a_n) does not have a unique limit.
(F_2-2) Then there exists L_1 and L_2 such that $\lim_{n\to\infty} a_n = L_1$, $\lim_{n\to\infty} a_n = L_2$ and $L_1 > L_2$. } (I_2-1)
(F_2-3) Let $\epsilon = \frac{L_1-L_2}{2} > 0$. By definition there exists an N such that if $n > N$ then $|L_1 - a_n| < \epsilon$ and $|L_2 - a_n| < \epsilon$.
(F_2-4) $|a_n - L_1| < \epsilon \Rightarrow a_n > L_1 - \epsilon = \frac{L_1+L_2}{2}$
$|a_n - L_2| < \epsilon \Rightarrow a_n < L_2 + \epsilon = \frac{L_1+L_2}{2}$ } $(I_2-2/3)$
(F_2-5) But then for $n > N$ we have that $a_n > \frac{L_1+L_2}{2}$ and that $a_n < \frac{L_1+L_2}{2}$ which is impossible. } $(I_2-4/5)$

(D_2-1) Toward a contradiction assume the sequence (a_n) does not have a unique limit.
(D_2-2) Then there exists L_1 and L_2 such that $\lim_{n\to\infty} a_n = L_1$, $\lim_{n\to\infty} a_n = L_2$ and $L_1 \neq L_2$.
(D_2-3) By definition for every $\epsilon > 0$ there exists a N such that if $n > N$ then $|L_1 - a_n| < \epsilon/2$ and $|L_2 - a_n| < \epsilon/2$.
(D_2-4) Consider $n > N$.
$|L_1 - L_2| = |L_1 - a_n + a_n - L_2|$
(D_2-5) $\leq |L_1 - a_n| + |a_n - L_2|$ via the triangle inequality
$= |L_1 - a_n| + |L_2 - a_n|$
(D_2-6) $< \epsilon/2 + \epsilon/2$
$= \epsilon$
(D_2-7) So $|L_1 - L_2| < \epsilon$. Since ϵ can be made arbitrarily small $L_1 = L_2$, contradicting the assumption that the limit was not unique.</td></tr>
</table>

Figure 1: If a sequence (a_n) has a limit, it is unique.

Unlike F_2, D_2 is not intrinsically an argument by contradiction. The contradiction was artificially added to proof D_2 to make D_2 and F_2 superficially similar. F_2 starts off assuming, toward a contradiction, that the limit is not unique (I_2-1 and F_2-1) and because of this we may choose two limits, L_1 and L_2 (I_2-1 and F_2-2). Although the assumption that $L_1 > L_2$ (F_2-2) does not explicitly appear in the informal argument, it is implied by the accompanying diagram. Next I_2-2 and I_2-3 argue that the ε-neighborhood around L_1 and L_2 can be made small enough to not overlap. In proof F_2, this "not overlapping," is achieved by choosing ε to be exactly half the distance between L_1 and L_2 (F_2-3), and then showing this choice of ε places a_n both above and below the midpoint (F_2-4) for all n sufficiently large. Finally, both Proof F_2 and the informal argument end by arguing that being in two places at once leads to a contradiction (F_2-5 and I_2-4/5). In the proof this is done formally by arguing

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that a term in the sequence, a_n, cannot be both above and below the mid point. Proof D_2 demonstrates that any two limits of a sequence can be made arbitrarily close to each other, and thus must be equal, but does not rely on the "two places at once" idea.

Results

A top-level view of the connections the participants made relative to our pre-agreed standards and hence, from our perspective, made normatively correct FBI-judgments for normatively correct reasons can be found in Table 1. As can be seen from the table, the first task was relatively unproblematic for the students in this study. The other two tasks were more difficult. Only 2 of the eight students met our minimum standard on both parts of task 2 and none of our students met our minimum standard on both parts of task 3.

These data point to mathematics majors' difficulties with FBI-comparison tasks. The ability to recognize the final product of formalization is crucial. Without this a student cannot recognize when the formalization process is complete and thus cannot effectively formalize.

Table 1: Student FBI-judgments relative to the minimum standard

	Task 1	Task 2	Task 3
Met criteria for D-I judgment	7/8	5/8	2/8
Met criteria for F-I judgment	8/8	3/8	1/8
Met both F-I and D-I criteria	7/8	2/8	0/8

A model of what students pay attention to when making FBI-judgments

In our analysis we identified four different aspects of arguments/proofs that students focused on when making FBI-comparisons. Two of the foci outlined below are adaptations of Pedemonte's (2007) structural distance and content distance constructs. The reframing of these constructs was necessary because the focus of Pedemonte's research is different from our own. The four foci of comparison are described below:

(1) Structural foci involve noticing global similarities and differences in which inferences follow from one another (i.e., structure). We conceptualize this as an adaptation of Pedemonte's (2007) notion of "structural distance" to a FBI-comparison context. In a broad sense structural foci can be seen as an attempt to evaluate what kind of relationship exists between the structure of an informal argument and the structure of a formal proof.

(2) Content foci involve noticing which specific elements (i.e., inferences, assumptions, data and claims) are present or not present within both an argument and proof. This can be seen as an attempt to evaluate the relationship between the content of an informal argument and the content of a formal proof. This focus can be seen as an adaptation of Pedemonte's (2007) notion of "content distance" to an FBI-comparison context.

(3) Methodological foci involve noticing the proof method used (e.g., contradiction, contrapositive, induction, construction, etc.) as well as the role this method plays in the proof.

(4) Holistic foci involve noticing similarities and differences in terms of goals, style purpose or overarching idea. These comparisons focus on proofs and arguments as a whole and overlook specific structural, content and methodological details.

Here we show that prioritizing one of the four foci of comparison in lieu of others has detrimental effects on students' ability to make informal to formal comparisons. It is important to note that these foci are not necessarily static. Some students may shift foci when moving to a different task.

Content foci. Next we discuss content foci. This involves paying attention to the assumptions and inferences within a proof/argument, but largely overlooking the roles and structure of these elements within a proof. Here the focus is localized to specific steps in the proof. In other words, the

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details of a proof are examined, but the bigger picture is ignored. We illustrate this focus with an excerpt of S8's work on Task 2. In the excerpt below S8 is asked to compare D_2 and F_2:

Int: Any… any other differences you can see? Looking at the proofs again?
S8: Umm… L_1 is defined to just not be equal to L_2 in proof D_3 and in proof F_3, they say L_1 is greater than L_2.
Int: I, I guess, … do you consider those significant differences? The ones that you mentioned?
S8: Yea, yea definitely. Those are significant differences.

The assumption that $L_1>L_2$ is not explicitly made in the I_2, however, it is implied by the accompanying diagram. Following the above excerpt S8 proceeded to use the fact that the $L_1>L_2$ assumption is part of proof D_2 and not F_2 as a justification for why the distractor proof is based on the informal argument while the formalization proof is not. In focusing in on a particular piece of content which is present in only one of the proofs he overlooks the bigger picture and consequently, makes an FBI-judgment that is in conflict with the normatively correct interpretation. Hence, we contend that prioritizing content in leu of structure is also insufficient for making normatively correct FBI-judgments.

S8's assessment is consistent with his content foci. He is looking for specific inferences that are present in the proofs in order to compare them to the informal argument. Thus, his expressed de facto conception of what it means for a formal proof to be based on an informal argument involves the formal proof using similar assumptions and similar inferences to the informal argument. Within this conception, a difference in assumptions used is sufficient evidence that a proof is not based on an informal argument.

Methodological foci. It is useful to note that our anticipation of methodological foci influenced our task design. Methodological foci were the motivation for making proof D_2 artificially a proof by contradiction. If we had not artificially made D_2 a proof by contradiction students may have concluded that D_2 was not based on I_2 solely based on the fact that I2 is a proof by contradiction without working to make other connections. Thus our task design intentionally discouraged surface level methodological foci. However, one participant did notice this feature of D2:

S7: Wait, what? … There is no point in this [D_2] being a proof by contradiction. That is completely redundant I could have just crossed this out here, "Assume it does not have a unique limit." You can cross that out.

Our task design intentionally prevented superficial methodology based assessments but left the tasks open to deeper assessments like the one in the excerpt above. Since only one of the participants noticed this feature of D_2 , we argue that artificially adding or removing particular methodologies from proofs has the potential to lead students to make incorrect FBI-judgments. That is, if we used a direct proof as a second distractor in place of F_2, we anticipate that the majority of students in our study would have incorrectly used "only one of these is a proof by contradiction" as a justification for why D_2 was based on I_2. This highlights the limitations of a strictly methodological foci.

Holistic foci. The final foci we discuss involves examination of holistic traits. The word trait here is construed broadly and may include attribute such as elegance, efficiency, style, pedagogical purpose or overarching idea. In short, this is intended to capture any treatment of a proof as more than the sum of its parts. Proofs have purposes and can be qualitatively compared to both each other and to the general genre of proof writing.

First we begin by discussing the work of S1 on task 1. The excerpt below begins after S1 reads proof D_1 (He has already read proof F_1).

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S1: I feel like D_1 was kind of lamer than the other one
Int: Lamer?
S1: This one [F_1] was a little prettier, it was… I mean over here, we had uh, we were using that it was… this [D_1] felt very… brute force

S1 treats the two proofs in task 1 as aesthetic entities and does not solely focus on the internal (line-by-line) workings of the proofs. He expresses the belief that elegant proofs are more desirable than brute force proofs and judges D_1 as less desirable. Later in his interview, when he was asked to compare F_1 and D_1, he discussed the two proofs relative to the genre of proof as a whole.

S1: Okay, what do they have in common? Clearly they have the goal in common, but the guy on the left, proof D_1, proof D_1 felt more like uh… I don't really know if there's an actual distinction in the math world between a "prove something" and a "show something," but if there was, this [D_1] definitely feels like, you know, just show that it's 0. But this [F_1] was like a really… this felt like it had more behind it here… whereas this [D_1] was like, let's just evaluate it and see where that takes us. Okay? Which is fair, you know? It just doesn't give you any insight into why that's the case.

S1 expresses the belief that proof D_1, does not provide any mathematical insight regarding why the result holds. It is simply an exercise in implementing well-established calculation techniques. Implicitly, he expresses that he often looks for what insights he can gained from presented proofs, in this case he did not find any.

However, one cannot effectively make FBI-judgments by focusing on holistic attributes of a proof in lieu of other attributes. For example, there may be multiple elegant arguments that justify a result. Thus, elegance alone is insufficient for making comparisons.

Multi-focus comparisons. In the previous subsections we argued that prioritizing only one of the foci of comparison in lieu of others was insufficient for making normatively correct based on judgments. In this section we illustrate what comparisons that utilize all four foci look like and how they may yield normatively correct FBI-judgments. To clarify we are arguing that balancing ones attention to these foci greatly increases the likelihood that a student consistently generates correct FBI-judgments but does not guarantee normatively correct judgments.

Below we examine S7's immediate reaction after reading proof F_2 for the first time:

Int: General impressions.
S7: F2 is just literally the proof version of I2…. Uh, so the idea behind this is that, okay if we are trying to show that this sequence has a unique limit, which we want to show that it can't have two limits. So we suppose there is two limits, basic proof by contradiction. So both are proofs by contradiction. And the contradiction occurs when epsilon is small enough. Here they show it intuitively but it's pretty clear from the picture that what they use was a number that's less then half way in between. Here [F_2] is that function, the average. So once we have the. It's not the average it's close enough so that it doesn't even reach the average. And that way the two have no overlap. And then by definition of sequence it should eventually get far enough that it's in this region always and once you get far enough it's in this region always but then it's therefore always in both these regions once it passes that specific end that we defined. And that's the contradiction. Which is what they said here. When we get small enough down we can't be in both but it has to be in both.

It is important to emphasize that S7 realizes that F_2 is based on I_2 before he is asked to make any kind of comparison. The above is simply his initial response. The part of the interview where he will be specifically asked FBI-questions occurred 30 minutes later. Also, he immediately jumps into the comparison when he states that both the proof and the informal argument have the same idea behind them (holistic foci). He then shifts to discussing how this idea manifests itself in terms of structure of both I_2 and F_2 (structural foci). He notes that both the proof and informal argument are necessarily arguments by contradiction with the contradiction in both cases being that you cannot be in two places at once (methodological foci). This is then related to the specifics of the proof, with being both above and below the midpoint of L_1 and L_2 corresponding to being in two places at once in the informal argument (content foci). S7 makes all four types of comparisons, does this without any specific prompting to make a comparison and relates the four types of comparison foci in his discussion.

We believe that the fact that S7 saw the relationship between I_2 and F_2 before he was asked to compare them to be particularly important. Mathematics is often discussed metaphorically as a language. Here S7 recognized that the informal argument and proof were metaphorically telling the same story. This is akin to being shown two paragraphs that tell the same story in two different languages, both of which one is fluent in. The fact that the same story has been presented twice, as well as the multitude of parallels between its two presentations is salient even without being asked to compare the two paragraphs.

On the other hand, if one is learning the second language and is asked the same question the comparison is very different. The comparison becomes an exercise in finding parallels between the words and phrases used, as well as, the order in which these appear. In this case one is likely to grasp onto only a fraction of the similarities and differences between the two paragraphs and make a determination based on only this subset. This is analogous to what we observed students doing when they prioritized one of the foci over others.

Discussion

This paper contributes to the literature on proof and proving both methodologically and theoretically. First, from the perspective of theory, this paper introduced a four-part model of the aspects of arguments/proofs students focus on when attempting to determine whether a particular argument is based on a particular proof. The components of this model are content foci, structural foci, methodological foci and holistic foci. We illustrated that comparisons where students prioritized one of these categories of comparison in lieu of others were prone to incorrect or incomplete conclusions regarding whether a proof was based on an informal argument. Furthermore failure to see the rich connections between informal arguments and proofs point to students having under developed conceptions of what if means for a proof to be based on an informal argument. These underdeveloped conceptions account for difficulties students have with generating proofs based on informal arguments (e.g., Zazkis et. al., in press) and in understanding the connections between informal arguments and proofs presented in lecture (e.g., Lew et al., 2014). These underdeveloped conceptions also account for some of students' resistance to generating informal arguments during proof production.

Methodologically, the triples method introduced in the study is a valuable research tool for those interested in research on the connections between informal arguments and formal proofs. Examining how students compare and contrast ready-made informal arguments and formal proofs provides valuable insights regarding what they notice when making FBI-judgments. In turn, what students' notice during FBI-judgments provides a valuable lens into how they conceptualize formalization and how they might view formalizing their own informal arguments. This method was able to reveal that students' conceptions of what it means for a proof to be the basis of an informal argument are not as rich as an expert conception—often only encompassing a fraction of the connections that exist between informal arguments and formal proofs.

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