MENTAL MATHEMATICS AND ENACTMENT OF SPECIFIC STRATEGIES: THE CASE OF SYSTEMS OF LINEAR EQUATIONS

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This study on the mental solving of systems of linear equations is part of a larger research program, aimed to gain a better understanding of the potential of mental mathematics activities with topics/objects other than numbers. Through outlining the study details and the activities engaged with, the paper reports on the variety of strategies developed for mentally solving systems of linear equations. The analysis pays attention to specificities and particularities of these strategies, focusing on their economical, tailored, and spontaneous nature to solve the tasks. In reflecting on the context of mental mathematics, the paper closes with discussions of the potential and richness of these strategies for exploring systems of linear equations.

Keywords: Algebra and Algebraic Thinking; Problem Solving; Instructional Activities and Practices

Context of the Study

To highlight the importance of teaching mental calculations, Thompson (1999) raises the following points: most calculations in adult life are made mentally; mental work develops insights into number system/number sense; mental work develops problem-solving skills; and mental work promotes success in later written calculations. These aspects stress the nonlocal character of doing mental mathematics with numbers, where the skills being developed extend to wider mathematical abilities and understandings. Indeed, diverse studies show the significant effect of mental mathematics practices with numbers: on students' problem solving skills (Butlen & Pézard, 1992; Schoen & Zweng, 1986), on the development of their number sense (Murphy, 2004; Heirdsfield & Cooper, 2004), on their paper-and-pencil skills (Butlen & Pézard, 1992) and on their estimation strategies (Heirdsfield & Cooper, 2004; Schoen & Zweng, 1986). There is thus an overall agreement, and across contexts, that the practice of mental mathematics with numbers enriches students' learning and mathematical written work about calculations and numbers: studies e.g. conducted in US (Schoen & Zweng, 1986), France (Butlen & Pézard, 1992), Japan (Reys & Nohda, 1994), and UK (Murphy, 2004; Thompson, 1999). However, there is much more. For Butlen and Pézard (1992), the practice of mental calculations can enable students to develop new and economical ways of solving arithmetic problems that traditional paper-and-pencil contexts rarely affords because these are often focused on techniques that are too time-consuming in a mental mathematics context. In a similar vein, Poirier (1990), in reviewing historical curricular documents where the practice of mental calculations was salient, underlines the fact that mental calculations have their own processes, which differ from regular written calculations. Others, like Murphy (2004) or Threlfall (2002, 2009), focus on the aliveness and on-the-spot nature of mental calculations, where strategies are developed and tailored for the problem at hand and often differ from the writing processes that are usually focused on in paper- and-pencil contexts: something I have also discussed at PME-NA-35 (see Proulx, 2013a).

This being so, as Rezat (2011) explains, most if not all research studies on mental mathematics have focused on numbers/arithmetic. However, mathematics involve much more than numbers, and are predominantly studied through paper-and-pencil activities. This rouses interest in studying (1) what doing mental mathematics with mathematical topics/objects other than numbers (e.g., algebra, functions, trigonometry) might contribute to mathematical reasoning and understanding of these topics/objects, as well as studying (2) the kinds of strategies and solving processes engaged in to solve mental mathematics tasks related to other mathematical topics/objects than numbers. This paper focuses on this latter interest in relation to systems of linear equations, that is, studying the

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nature and specificities of the strategies developed through solving systems of linear equations tasks in a mental mathematics environment.

Defining Mental Mathematics

Because most work on mental mathematics is on numbers (often referred to as mental *arithmetic* or *calculations*), no definition of mental mathematics appears in the literature. For Thompson (2009), mental calculations represent a *subset* of mental mathematics; however he does not offer any definition of mental mathematics. This said, even if thought of in terms of numbers and mental arithmetic, definitions about mental calculations can be adapted to other mathematical objects/topics to help define mental mathematics. Building on Hazekamp (1986), who offers a definition that summarizes what is generally considered by mental calculations, one tentative definition is: Mental mathematics is the solving of mathematical tasks through mental processes without paper and pencil or other computational (material) aids.

In order to help develop a finer understanding of what is meant by mental mathematics, various dimensions about mental strategies with numbers are found in the literature (e.g. Butlen & Pézard, 1992, 2000; Kahane, 2003) and are adaptable to other objects/topics than numbers. One of these dimensions concerns *reasoned computations*, implying the elaboration of personal strategies, often nonstandard and adapted to the problem, versus *automatized computations*, which implies access to an immediate result through the use of known facts or memorized procedures. An example of this could be, for area, between using the formula $\frac{D\times d}{2}$ to find the area of the rhombus versus cutting the figure into triangles to find or compare the area. A second set of dimensions concerns *approximate computations*, based on estimation and approximation to gain an order of magnitude for the answer, versus the mental application of an algorithm or a fact to obtain an exact answer. An example for trigonometry could be between using the fact that $sin30° = \frac{1}{2}$ versus establishing a visual order of magnitude that the opposite side of a 30° angle enters approximately twice in the hypotenuse. A third dimension concerns *rapid computations*, which require quick execution to find the answer. Often criticized because it is perceived as a speed exercise detrimental to sense-making, it can also be seen as helping to develop new solving methods because it forces the solver, in trying to be economical, to abandon methods that may be slower (e.g. standard procedures) or less efficient for completing the task (e.g. one-on- one counting). In the case of algebra, an example could be the development of a *global reading* of an equation like $x + \frac{x}{4} = \frac{4}{x} + 6$ giving $x = 6$, avoiding numerous algebraic manipulations in order to isolate *x* (Bednarz & Janvier, 1992). These dimensions illustrate possible entries for solving mental mathematics tasks. In the case of systems of linear equations, what these dimensions represent is something probed into in this paper. In addition to having value for refining what mental mathematics can mean, these dimensions have value for data analysis and are reinvested in the subsequent analysis of strategies developed by solvers for systems of linear equations.

Methodological Considerations

This study is part of a larger research program that focuses on studying the nature of the mathematical activity in which solvers engage with through working on mental mathematics with objects/topics other than numbers. This is probed through (multiple) case studies that take place in educative contexts designed for the study (classroom settings/in-service education activities), where participants are asked to solve a variety of tasks. The reported study is one of these case studies, from a day-long session with 12 secondary-level teachers (Grade 9 to 12). The general organization took the following structure: (1) a task is offered to the group in writing on the board; (2) participants have approximately 15 seconds to solve the task; (3) at the signal, participants are asked to write their answers; (4) strategies are shared in plenary. The data comes from the strategies orally

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explained by participants, recorded in note form by two research assistants, who collected and compared their notes to produce more substantial information about the strategies developed. The session was also video-recorded, which allowed the research team to return to the tapes to enlarge on the notes and analyse the strategies in depth.

In this study, teachers are regarded as problem solvers; as would any participant solving the given tasks in this project. The decision to work with teachers is methodologically important. Indeed, these teachers are not novice solvers of systems of linear equations and thus are not in a new solving context or familiarization with the topic. This enables them to “enter” into tasks and attempt to solve them, giving access to their strategies and mathematical activity. This could be otherwise with participants who are newcomers to the topic, as they could experience significant difficulties with systems of linear equations and possibly would not be able to “enter” into the tasks for solving them. In that sense, the intention in this study is not to offer prescriptions for practice or to show how an approach through mental mathematics is “better” for learning about systems of linear equations than another focused on paper and pencil. The intention is neither to trace parallels between what these teachers do and what could happen with secondary-level *students* in a regular classroom. The analysis is focused, along the lines explained by Douady (1994), on the nature, meaning and functionality of the strategies developed in this mental mathematics context in order to analyse the mathematical activity engaged with and study the specificities of the strategies deployed for solving systems of linear equations tasks.

Findings

The tasks focused on for this paper consisted mainly in finding the solution, the intersection point, of a system of two linear equations (given algebraically), and then drawing the coordinate point on a sheet of paper displaying a Cartesian graph (with $y = x$ drawn as a reference line). Without being too caricatural, in a paper-and-pencil context the participants (as they explained during the session) would have resorted to algebraic manipulations and known algebraic strategies (comparison, elimination, substitution methods) to find these intersection points and then draw them on the graph. While solving the tasks, some participants opted for these algebraic methods, but in most cases they could not do this as the burden of algebraically manipulating without paper-and-pencil support became too important, and they had to develop alternative ways of solving. Below, I report on these alternative strategies to give a sense of their nature and analyse the mathematical activity involved. I first focus on the strategies given for solving the system of linear equations: “$y = x$ and $y = {}^{-}x + 2$”. This is followed by other strategies developed for similar tasks, focusing again on their specificities for solving.

Strategies developed for Task 1 (“$y = x$ and $y = {}^{-}x + 2$”)

Strategy 1: The role/influence of parameters. To solve this system, the participant focused on the fact that both lines would normally cross at (0,0). However, because the second equation was not “$y = {}^{-}x$” and had a y-intercept of 2, the answer was elevated by 2 on the y-axis, giving (0,2) as an answer. The focus on the parameter (y-intercept) played an important role in determining what and where the answer would be in the graph. This said, the answer (0,2) was not seen as the *y*-intercept, but mainly as an elevation of the point (0,0) toward (0,2); it ends up being the same, but at the time it was not thought of in these terms, as the participant expressed.

Strategy 2: Finding a “line” of possible solutions. The participant drew a vertical line at x=1, explaining that he did not have enough time to find the exact value of y, but that the solution was on that line because x=1 gave the same answer for both equations. Of interest is that substituting x=1 also gives the value for y. But in his algebraic manipulations for finding the value that gave the same answer for both equations, his focus was on finding a common x that gave the same answer for both

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equations ($x = ?$ and $^{-}x + 2 = ?$) and not on finding the value of *y*, even if it is the same. Both ventures were seen as separate. The first venture (focus on *x*) gave an infinite number of solutions within a restricted domain of *x*=1. He did not have enough time to look for *y*.

Strategy 3: Visualizing in the graph. The participant mentioned having found approximately where the point would be, in the 1st quadrant, by visualising the lines as one that crosses the first quadrant in the middle (y=x) and the other going through the 1st quadrant as well ($y = {}^{-}x + 2$, with a negative slope and starting from 2 on the y-axis), both intersecting on $y = x$ and in the 1st quadrant. Visualising the lines played an important role to position where the intersection point would be.

Strategy 4: Visualizing the lines with objects. Having pencils and pens on her table (for drawing the solution afterwards), the participant imagined them as lines in the graph and saw, as did the participant in Strategy 3, that the intersection point was in the 1st quadrant. It is through "seeing" the graph that the solution was developed, as the participant explained.

Strategy 5: Finding the right quadrant. For this participant, the first step was to realize that the solution would be in the 1st quadrant, because of the equations of the lines: one that splits the 1st quadrant in half and the other that goes through it (without being precise in exactly how). The point was not placed precisely in the graph (close to the *x*-axis, a little to the right of the origin), and the participant knew this, but the focus was on finding the right quadrant for the intersection.

Strategy 6: The y-intercept as a focus. Similar to Strategy 1, the *y*-intercept played a role in determining the intersection point. This time, with the difference that the *y*-intercept did not influence a previously obtained answer (as in Strategy 1 where the point (0,0) was elevated by 2), but influenced the fact that the solution had to be "in that area" of the *y*-intercept because it played a role in the position of the line $y = {}^{-}x + 2$ in the graph. Hence, the solution was placed at (0,2), mainly because of time constraints and because an answer had to be given. Here again, even if it is the same coordinate points, (0,2) is not thought of as the *y*-intercept for the participant (if it had been, it would have been discarded, because the participant knew that the other line did not have 2 as *y*-intercept). It was seen as a possible intersection point where both lines would cross each other.

Strategy 7: Visualizing the lines with gestures. Similar to Strategy 4, a participant imagined placing and crossing hands to represent the slopes of the lines, which is a common teaching practice. This gave an idea of where the intersection point would be. He considered the point to be in the 1st quadrant and saw that the intersection point could be in the "middle" of the crossing hands (not necessarily realizing that the point was on *y*=*x*, with the same value for *x* and *y*). Here too, the participant explained that it is through seeing the lines that the solution was developed.

Strategy 8: Trial and error. The participant attempted some numerical solutions (however not of the form *x*=*y*), but because it led nowhere he was unable to place any point of intersection (albeit knowing that there was one because both lines did not share the same slope).

In the following section, the above strategies are grouped and discussed in relation to their similarities, drawing out their specificities related to the context of mental mathematics. Since a strategy's attributes can be related to diverse groups, it can be placed in more than one group.

Discussion of strategies developed for Task 1 ("$y = x$ and $y = {}^{-}x + 2$")

Order of magnitude strategies: Strategies 2, 3, 4, 5 and 7. For these solutions, the focus is on having a good idea of where the solution is in the graph, of what is happening in the system of equations. These solutions can thus be related to the *approximate calculation* dimension. The focus is not on finding an exact answer, because time i s an issue in the mental mathematics context. This forces an analysis of the system to obtain an order of magnitude, an approximation, of where the solution would be: whether by focusing on the value of *x* as in Strategy 2, on the fact that the solution is on the line *y*=*x* as in Strategy 3, on gaining a visual idea of where the lines intersect as in Strategies 4 and 7, or of knowing in which quadrant the solution is as in Strategy 5. The need to develop an order of magnitude appears to derive from the mental mathematics context, a strategy quite removed from the algebraic manipulations that aim to find the exact solution to the system. The specificity of these solutions underlines important aspects, mainly visualizing the system and understanding what is happening. The focus on x=1 in Strategy 2 and on x=y in Strategy 3 are exam ples of how this focus on the order of magnitude offers significant information about the system, because, even if it may seem obvious, the value of x and the value of y need to be the same for finding a solution that satisfies both equations, one of them requesting that x=y! The same is true for the more visual strategies where lines are positioned, because one can directly see that there is an intersection point (something important to know about) and that the links between both algebraic and graphical representations are salient. Hence, one gets an idea of the system, how it functions, and what its possibilities are. It is in this sense that these strategies appear as specific for mental mathematics: not that they are better or worse, but simply different, provoked by the context of solving and offering another way into solving the system that a paper-and-pencil context does not necessarily afford.

Study of the equation: Strategies 1 and 6. Even if they give an answer that is mathematically inadequate, these strategies have something interesting to offer and can be related to the *rapid computation* dimension for a *global reading* of the equations. In particular, they are quite far removed from the usual algebraic manipulations that one would normally plunge into to solve the system. When solving algebraically in a paper-and-pencil context, the presence of the 2 in the second equation, the y-intercept, is not given much consideration or seen as important as it is simply a “2”, the number “2”, that is manipulated to find the value of x or y for solving the system. In algebraic manipulations, this “2” is not related to a “2” as the y-intercept, and the answer obtained is not considered as being influenced by this “2”; any more than would the negative sign affecting the x in the second equation (in algebraic manipulations, this negative sign affecting an x does not signify a “negative” slope). But, in Strategies 1 and 6, this “2” *is* the y-intercept and not simply a number to manipulate. Hence, consideration of the “2” is not related to the mechanics of manipulating algebraically, but for reflecting and thinking of the answer, the point of intersection of both lines. Thus, in these two strategies, one finds that the equation is studied, its attributes are considered and evaluated for finding something in it, for reading it so that it speaks differently than just algebraic symbols that need to be mingled. In this sense, these strategies call attention to the “-” sign affecting the x, to the “2” in the second equation, that makes the equation different from one that does not have the “2” (e.g. $y={}^{-}x$). These strategies offer a focus on differences in equations that make a difference for determining the solution.

Because of time constraints, the effect of the “2” made the participants decide on an intersection that was about that “2”, but the specificity of these strategies is not about the answer obtained or the consideration of the “2”, but rather about the effect that this “2” had on the answer: the fact that it influenced and played a role in the answer (e.g. 2 more, 2 higher). The specificity of Strategies 1 and 6 concerns the focus on the *effect* of the parameters, here the y-intercept, on the solution to the system: something of lesser interest in algebraic paper-and-pencil manipulations.

Focus on numerical points: Strategies 2 and 8. These strategies focus on exact values for the solution by attempting to substitute possible answers to satisfy the system: being a hybrid between automatized and the opposite of *approximate computations*. It can be said that these strategies are elementary, as they are only an attempt to try out possible answers in a trial-and- error, unsystematic venture. However, their interest lies not in what they are not, but mainly in what they can be and can offer: that the solution needs to satisfy both equations simultaneously to be a solution to the system. Indeed, if there were only one equation to satisfy, the numbers attempted would have given a solution, and this would be it (e.g. x=2 in $y={}^{-}x+2$ would give y=0 and this would be satisfactory for an answer to this equation). But, in the case of a system of equations, the solution needs to satisfy both equations simultaneously to be its solution. Hence both x and y need to be solution of both equations.

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The x=1 answer in Strategy 2 illustrates this, as x=1 is explained as a solution for both equations, meaning that it gives the same answer in each equation. (Again, here the focus of the participant was on finding x first, and time did not allow for finding y, even if the "answer" and the "y" were the same). The next step would have been to find the value of y that was the same for both equations, which would satisfy both simultaneously. The same is true for Strategy 8, as the solutions attempted (here coordinate points) was intended to test a value for x and for y at the same time in both equations and to see if both satisfied the equations. The specificity of these strategies lies in the fact that the mental mathematics context provoked a need to find an answer, an x and a y, that satisfied both equations. In an algebraic manipulation context, this intention to find values that satisfy both equations simultaneously is often hidden behind the mechanical manipulations, and if it emerges it does so at the end of the process when establishing the values of x and of y that are solution to the system. It is in this sense that the trial and error, focused on numerical points, appears as specific in this mental mathematics context.

Discussion of a variety of other strategies developed for other tasks

In what follows, I discuss two other strategies developed to solve similar tasks. In detailing these strategies, I explore their specificities and what they focus on for solving.

Visualizing the lines in the graph through studying the equation. When solving the system "y=3x+1 and y=7x", one strategy was to analyse where the lines would be in the graph and then consider their intersection; related to a *global reading* dimension of the system. Thus the participant explained that y=7x passed by (0,0) and is quite inclined, whereas y=3x+1 "starts at 1", is less inclined, and thus crosses y=7x at a y greater than 1 and on the right of the y-axis (note that the analysis is made only in the 1st quadrant, as this participant knew where the lines would intersect from visualizing both lines, whereas an algebra-equation analysis would not offer the same information about the 1st quadrant). Another similar strategy was also related to the inclination of the lines, where the y=7x was seen much more inclined than the y=3x+1, leading to a value in x being between 0 and 1. Also, the value of y was seen as greater than that in x because one of the two lines, the y=3x+1, had a y-intercept of 1 (and a positive slope). In both cases, the analysis of the line is precise, focusing on aspects of the equation that gives information about the lines in order to visualize them for subsequently finding the solution. Understanding that the solution is in the 1st quadrant because of the inclinations of the lines, that the value of x is between 0 and 1, that the value of y is higher than 1, and so forth, is not a necessary part of the algebraic manipulating process, as these facts have little influence on the manipulations needed to obtain the solution. But in this case, in the mental mathematics context, these specific aspects are provoked in the strategies. And similar visualizations of the line in the graph, through studying and analyzing the equation, were made for other tasks. E.g., with the line "$y = x + 10$", one participant said that it was parallel to "$y = x$" but higher because of its y-intercept. Another example is with the system "$y = \sqrt{8}x + 5$ and $-\sqrt{18}x + 3y = 9$", where one participant approximated the value of the slope of $\sqrt{8}$ as being close to $\sqrt{9}$ and thus close to 3 with 5 as the y-intercept. For the second one, a mistake was made in relation to the sign of the slope, which was seen as positive rather than negative, but the participant considered the y-intercept as being of 3 (from dividing 9 by 3), and thus the solution in y as being between 3 and 5 in the 2nd quadrant. This analysis is quite impressive. For this participant, the intersection could not be lower than 3 in y, because it would be in the 1st quadrant, which is impossible because in that quadrant the other equation "starts" at 5 in y (its y-intercept being at 5). Thus by being a negative slope that has a y-intercept of 3, it had to intersect in the 2nd quadrant and between 3 and 5. These strategies represent specific ways to manage solving the system, performing a fine analysis of the equation combined with a visualization of the lines for solving it. It offers a specific way into the system, visualizing it, but mostly understanding how it works and where and how the solution can be. The specific analysis of the equation renders some "information" in the equation as significant to solve the system: something un-usual in a paper-and-pencil algebraic manipulations context.

Approximate algebraic manipulations. When asked to solve the system "$4x + y = 10$ and $x - 2y = {}^{-}6$", some participants opted for the elimination method by quadrupling the second equation, transforming it from $x - 2y = {}^{-}6$ to $4x + 8y = {}^{-}24$, and then subtracting it from the first (and obtaining$9y = 24$). Because of time constraints, the value obtained for y was said to be around 5; which can be related to the *approximate computation* dimension. Then, substituting the value of *y* in the first equation led to a value for *x* of around 1, for a coordinate point being about (1,5). Obviously, the degree of errors in regard to the solution is significant, because there is a first approximation for *y*, and then one for *x* based on that *y*. But of interest in this strategy is the approximation of the values in an algebraic system. This offers an order of magnitude for both values even if they were obtained through what is often seen as a precise strategy, that is, the algebraic route of elimination. Whereas earlier solutions focused on gaining an order of magnitude for situating the coordinate points in the graph, here the order of magnitude is in relation to algebraic manipulations: quite different from what is usually seen in an algebraic manipulation paper-and-pencil context.

Final remarks

These strategies reported illustrate some specificities of ways of engaging with the task in this context of mental mathematics. They are no doubt provoked by time constraints and because notes cannot be taken in writing, but this is the context of mental mathematics and it promotes these kinds of specific entries. It creates a need to grab something, to draw out aspects of significance for solving the problem. In addition to their specificities, these strategies have important potential for understanding or developing meanings about solving systems of linear equations, away from mechanical treatment and in relation e.g. to considerations of what is going on in the equations themselves, and how they behave. The reported "analyses/studies of the equations" represent such possibilities/potential for understanding systems of linear equations. Even if these strategies are not always optimal or do not render correct answers (precise or not), they offer ways of understanding what a system of equations is, ways that are different from what is usually done with algebraic manipulation strategies. These strategies are specific in the sense that they offer a way into the systems, an analysis of it, through drawing out particular aspects. Whether this is through analyzing the equations to gather significant information on the system, through having an order of magnitude of the solution or the system itself, through attempting to satisfy simultaneously both equations of the system, all this offers a specific way of engaging with solving systems of linear equations.

These results are promising. What comes out of this current work, and that on solving algebra equations (e.g. Proulx, 2013b) and operations on functions (e.g. Proulx, 2013a) in mental mathematics contexts, is the emergence of unusual ways of tackling and working on these mathematical topics, bringing forth varieties of strategies that focus on aspects not usually engaged in or focused on (here e.g. the *y*-intercept, the order of magnitude, the boundaries where the value in *x* and *y* could be). These strategies are creative and exploratory, and in this sense they suggest *extending* what can be done with these topics: not in terms of the tasks offered, but in terms of the meaning to be given to the topic itself. These strategies bring us elsewhere, focusing on different aspects and on other ways of solving. It is in this sense that they are specific, as is argued similarly for mental calculation on numbers, where they differ from the usual ways of solving and enable a focus on other aspects for solving. Clearly, more is to be studied and researched, but the focus on extending the scope of solving for these topics promise great value for mathematics teaching and learning.

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