JAKE'S CONCEPTUAL OPERATIONS IN MULTIPLICATIVE TASKS: FOCUS ON NUMBER CHOICE

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This case study examined how a teacher's choice of numbers used in tasks designed to foster students' construction of a scheme for reasoning in multiplicative situations may afford or constrain their progression. This scheme, multiplicative double counting (mDC) is considered a significant conceptual leap from reasoning additively with units of one (1s) and composite units. A researcher-teacher's work with Jake allowed us to center on his gradual cognitive advance as different numbers chosen for the unit rate in problems (e.g., 5 cubes-per-tower) were used in the context of the Please Go and Bring for Me platform task. Our findings show that a child's use of an evolving scheme may initially depend on the numbers used in the task. We discuss the key recognitions that (a) a new way of operating does not evolve in a "once-and-for-all" way for all numbers and (b) the support our study provides for Pirie and Kieren's core notion of folding-back.

Keywords: Number Concepts and Operations; Elementary School Education; Teacher Knowledge

Introduction

In recent years, a growing body of research has been focused on various aspects involved in children's transition from additive to multiplicative reasoning (Empson& Turner, 2006; Hackenberg & Tillema, 2009; Sherin & Fuson, 2005; Tzur et al., 2013; Verschaffel, Greer, & DeCorte, 2007). However, unlike in other mathematical areas, such as numbers chosen for addition and subtraction tasks (Fuson, 1992), how a teacher may consider the use of numbers in tasks designed to help students overcome the conceptual leaps involved in progressing from additive to multiplicative reasoning has received little attention. To embark upon this lacuna, our study addressed the problem: How may specific numbers a teacher uses in tasks for promoting students' advance from additive to multiplicative ways of operating with/on different types of units (1s, composite), afford or constrain a child's conceptual progression when solving multiplicative problem situations? In particular, this paper examines such affordances and constraints as a child begins the transition from additive reasoning to the first scheme in which she or he coordinates operations on two types of composite units—the multiplicative double counting (mDC) scheme.

Conceptual Framework

A constructivist perspective on knowing and learning (Piaget, 1985) underlies this study. Specifically, we drew on von Glasersfeld's (1995) construct of scheme as a three-part mental structure: (a) situation, a recognition template into which a learner assimilates a problem situation (or task), that triggers her or his goal; (b) a mental activity the mind carries out to accomplish that goal; and (c) a result the learner expects to follow from the activity. The situation part includes recognition of and bringing forth of objects, such as numbers, upon which the mental activity operates.

Working within such a perspective, Steffe's (1992) seminal work contributed to distinguishing between additive and multiplicative schemes. He proposed to focus on the units on which one is operating mentally (1s or composite units) and the operations a learner uses that underlie her or his performance when solving tasks. He thus distinguished additive from multiplicative reasoning not on the basis of observable behaviors ("strategies") the child uses in and of themselves, but on the basis of inferences into what mental operations on units operations could give rise to those behaviors. Specifically, he asserted that multiplicative schemes involve a coordination of composite units in

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which the child distributes the items of one composite unit (e.g., towers made of 5 cubes each) over items of another composite unit (e.g., a compilation of 3 towers). Such a coordination may be manifested in the use of some figural items in place of the objects alluded to in the task (e.g., a finger standing for one tower) and later in the abstract, double counting of two sequences of composite units (e.g., first tower is 5 cubes, second-is 10, third-is-15). Tzur et al. (2013) recently proposed a developmental pathway of six schemes through which children's multiplicative reasoning may progress; this study focuses on the use of numbers to promote the transition from additive reasoning to the first scheme in the progression.

This study also drew on Pirie & Kieren's (1994) constructivist stance on the non-linear growth in learners' mathematical understandings, particularly the key construct of folding-back. Rooted in studies of children in other areas (e.g., fractions), they showed that an ordinary path to higher-level understanding (outer layers in their model) might include frequent 'regresses' to the use of lower-level, previously constructed understandings. Our tasks were designed to promote students' transition to the first of six schemes, multiplicative double counting (mDC), while allowing the research team to analyze how numbers chosen for these tasks would possibly bring about folding-back and upward shifts in the units/operations a child may use.

Methodology

The case study reported in this paper was part of a larger constructivist teaching experiment (Cobb & Steffe, 1983) we have conducted with four 4th graders identified by their western US school as requiring intervention in mathematics based on state assessments and classroom teacher recommendations. The two first authors conducted the video recorded teaching episodes with each child individually. This paper analyzes data from their work with one student, Jake (pseudonym), twice a week, from October through December of 2014, around 30-45 minutes each episode. The second author (Nina, pseudonym) served as the researcher-teacher in those episodes, as Jake was accustomed to working with her as the school intervention teacher.

All teaching episodes engaged students in playing the task-generating game of Please Go and Bring for Me (PGBM), which Tzur et al. (2013) described in detail. In a nutshell, PGBM is a turn-taking game played in pairs, with one's peers and/or the teacher. Each turn, a "Sender" asks a "Bringer" to build and bring back from a box containing individual cubes a compilation of same-size towers, one tower at a time (e.g., 3 towers, 5 cubes in each). Once all towers were brought to the Sender's satisfaction, she or he asks the bringer four questions (in our work – those are written on a poster to promote students' use of full sentences and explicit mention of units): (a) How many towers did you bring (emphasizes number of composite units)? (b) How many cubes are in each tower (emphasizes unit rate – number of 1s in each composite unit)? (c) How many cubes are in all the towers? (d) How did you figure this [total of 1s] out? (Last two questions emphasize operations child used to figure out number of 1s in the entire compilation of composite units.) Similarly, the poster included 'answer-starters' that enabled the bringer to express her or his answers as full sentences (e.g., "I brought __ towers"). Initially, the teacher constrains the game so children can only use particular numbers of cubes per tower (e.g., 2 or 5) and of towers in all (e.g., up to 6), while also asking children to use different numbers for each kind (e.g., a sender cannot ask to bring 5 towers of 5 cubes each).

Our line-by-line retrospective analysis of video records, transcripts, and researcher field notes taken during each episode focuses on the first two teaching episodes with Jake. Focusing on his initial transition to multiplicative reasoning serves the purpose of 'zooming in' on the interplay between his ways of operating and the numbers chosen in each task. The two first authors conducted ongoing analysis following each episode; the entire team of authors then conducted the line-by-line analysis of the four segments presented in the next section.

Results

To study how numbers chosen for tasks may afford and constrain a child's operations with different types of units as the teacher promotes Jake's transition to the mDC scheme, this section includes analysis of four data Excerpts. In the first episode (Excerpts 1 & 2), Jake worked with a peer on PGBM tasks constrained to unit rate of 2 or 5 and a number of composite units up to 5. In the later episodes (Excerpts 3 & 4), Jake played a bringer role with Nina as sender, with tasks allowing both unit rates and number of composite units to be 2, 3, 4, 5, or 6.

Starting Point: Less than Five Composite Units, Unit Rates of 2 or 5

Excerpt 1 shows data from Jake's first turn as a bringer, after he had produced (from single cubes), brought 4 towers of 5 cubes each, and properly responded to the first two questions (SS stands for Student-Sender).

Excerpt 1

SS: How many cubes did you bring in all?

Jake: (Glances at the towers for 1 second, then says) I brought … (uses his left hand to tap five times on the palm of his right hand); I brought 20 cubes altogether.

SS: How did you figure this out?

Jake: I figured this out by counting.

Nina (Teacher): How did you count?

Jake: I counted by 5s[1].

Nina: How did you know to stop counting?

Jake: Cause if you don't, cause if you can't … (reaches with his left hand and brushes over the towers that the SS is holding)

Nina: I can count by 5s too: 5, 10,15, 20, 25,30 ... So how did you know to stop at 20?

Jake: (Looks at the towers that the student sender is holding) It's … because that … (turns his head away from the towers for five seconds, then turns his gaze back on the towers). It's because I only brought 4 towers.

Nina: So you knew to stop because ... ? How did you know you had counted the four towers, Jake? I agree that you only brought 4 towers. How did you know that 20 was 4 towers.

Jake: I counted on my fingers.

Nina: Can you show me?

Jake: (Holds up his right hand, then folds his index finger while stating) 5; (folds middle finger) 10; (folds ring finger) 15; (folds pinkie finger) 20! (Body indicates, "I am done").

Excerpt 1 provides a glimpse into Jake's mental operations while solving a problem with 'easy number' unit rate 5. Jake first reestablished the number of 1s that constituted each composite unit (five palm taps). That is, he seemed to have created a figural mental template of the size of every composite unit. Because counting by 5s was within his capacity, his count of the accrual—which he demonstrated to the teacher, indicated a purposeful method of keeping track of the number of composite units so he would know to stop at four towers. His ability to perform such a purposeful action did not yet seem to support expressing how he did it. Rather, when prompted to explain how he knew to stop at 20, he initiated a shift to a different figural re-presentation, by using the fingers of his right hand to represent the composite units in the situation (towers) and his number sequence (by five) to re-present the accruing 1s (cubes). This seemed to assist his growing anticipation of the link between coordinated actions taken to figure out a progressive total and the effect of stopping the count of 1s when reaching the number of composite units (which fits his statement, "I brought 4 towers"). Jake, when operating mentally on a unit rate that is a known counting sequence (5s), could

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both initiate and complete the coordinated, goal-directed mental activity involved in multiplicative double counting.

It is important to note the purpose of the teacher's interventions (e.g., "how did you count"). A child may initiate and carry out goal-directed actions while not being aware of his own actions, let alone the steps he took to monitor those actions. By asking Jake to explain, she thus attempted to orient his reflection on and awareness of his own purposeful actions (e.g., he did keep track of the composite units). To explain his strategy, Jake used fingers, which the teacher intended as a means to promote two critical reflections in constructing the mDC scheme. First, she focused Jake's attention on a specific aspect of his coordinated counting—monitoring accrual of the composite units. Second, she simultaneously focused his attention on the key in his stoppage monitoring.

In the following PGBM task in which Jake played the bringer, he had to figure out and explain how many cubes are in 5 towers with 2 cubes each. Again, Nina pressed for his explanation of his solution, as his response (10) came after quietly nodding his head five times but not using his hands/fingers. Excerpt 2 shows he re-used the previous way of explaining.

Excerpt 2: Finding the total number of cubes in 5 towers of 2 cubes each

Nina: Can you show me?
Jake: Like this … (holds up right hand, folds down his index finger while saying) 2; (folds his middle finger) 4; (folds his ring finger) 6; (folds his pinkie) 8; (folds his thumb and concludes) 10.
Nina: So, again, each one of your fingers … this [seems] similar to something else you did. Each one of your fingers, you put them down: 2, 4, 6, 8, 10 (paraphrases J's motions and utterances); so you stopped here (wiggles her thumb); Why?
Jake: Its because I only brought 5 towers.

Excerpt 2 provides further evidence to the evolving regularity in Jake's ways of operating as well as the teacher's involvement in that process. Freed from a mental focus on the accrual of 1s in the sequence of multiples of 2, he initiated and completed a coordinated count of figural composite units(5 towers) through nodding his head and later through using fingers. He seemed to anticipate the need to coordinate two accruing number sequences: composite units (towers) with unit rate (cubes distributed over each of the towers). Similarly, when prompted to explain his thinking, Jake used the fingers of his right hand to re-present each composite unit in the sequence of 5 towers while keeping track of accrual of 1s via his number sequence (by 2s) to ten. Considering Excerpts 1 & 2 combined, Jake seemed to have established at least an enactive anticipation of multiplicative double counting as a means to accomplish his goal.

Starting Point: Composite Units of Less Than 5, Unit Rates of 4 and 6

Excerpts 3 & 4 present Jake's work two days later, with Nina serving as the sender. Based on his facility with double counting in tasks with unit rates of 5 or 2, she chose to send him to bring 3 towers of 4 cubes each—slightly harder numbers (for him) that he could still work out by using each hand separately. Excerpt 3 starts after Nina asked Jake how many cubes he brought in all.

Excerpt 3

Jake: I brought 11 cubes altogether.
Nina: (Lays the towers down on the table closer to Jake) Can you double check?
Jake: (Takes apart each tower into its individual cubes while counting them out loud but without keeping them as distinct groups) 1, 2, 3, 4,5,6,7,8,9,10,11,12!
Nina: So how many [cubes] did you bring altogether?

Jake: 12.
Nina: I want to know how you figured that out (she takes 4 cubes and reassemble them into a tower.
Jake: (Reassembles another tower, albeit from 5 cubes.)
Nina: (Reassembles another tower from the remaining 3 cubes, and asks him to give her a cube so she can make a tower of 4.) Jake, when you counted the first time, I saw you use your fingers (she folds her left hand's fingers to emulate his motions). Can you tell me what you were doing?
Jake: (Folds down 4 fingers on his left hand, then folds down a finger on his right hand; repeats the process for the second and third towers, while left-hand fingers seem to stand for 1s and right-hand fingers for composite units/towers.)
Nina: And why did you do this (replicates Jake's motions)?
Jake: Each finger was a tower.
Nina: Each finger was a tower? So can you show me again?
Jake: (Holds up his right hand with the palm face up) Each finger was a tower. (Folds down three fingers on his right hand, one at a time, while saying) That's the first tower, that's the second tower, and that's the third tower. (He then adjusts his hand motion and folds down four fingers on his left hand, one at a time, while saying)If we add them all up we go 1(folds down a finger),2 (folds down a second finger),3 (folds down a third finger), 4 (folds down a fourth finger),5 (looks at his hand as if having an 'oops' experience). (After 2 seconds, he opens his hand, says "No," then raises three fingers again on his right hand and says)No; this is three towers and you [have to] count by fours. He uses his left hand to put down one of the fingers on his right hand and says "Four." Then he raises his left hand, counts four fingers, and says "5-6-7-8." He then uses his left hand to fold another finger down on his right hand, then opens his left hand again and says "9-10-11-12."(Then, folds the third finger on his right hand.)
Nina: And you knew to stop at 12 because these were like the towers (touches the fourth finger on *his* right hand and this was like the 12th …
Jake: (Completes her sentence) … Tower
Nina: (Inquiring about the unit type) The 12th what?
Jake: Tower
Nina: (Places her hand on the real towers and says) How many towers do you have here?
Jake: (Corrects himself in response to her prompt) That was the last, third tower.
Nina: It was the last, third tower; but it was the 12th what?
Jake: Cube

Excerpt 3 provides further evidence of the role numbers in task played in his evolving way of operating, as well as the teacher's involvement in that process. Challenging him to solve a problem with 4 as the unit rate, Jake's mental system was no longer freed from focusing on the accrual of 1s. Thus, he folded back to a focus on 1s (single cubes) that composed each unit. Taking apart all 3 towers without any allusion to their grouping indicated his need to individualize units—no longer reasoning multiplicatively. Yet, when prompted to explain, Jake made several attempts to coordinate the use of his fingers to re-present, on each hand, a different type of unit (cubes on the left hand, towers on the right). In his first, unsuccessful attempt, he identified the right fingers as towers, but began counting them as cubes. In an "oops" moment, he realized that counting the figural tower re-presentations on his right hand was not moving toward his goal and started over. On his second attempt, Jake was able to accurately use his fingers to keep track of the unit rate of 4. He began his count not from 1 (cube) but from the first multiple of 4, indicating the coordination of a first

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composite unit with the numerical value of the unit rate. Due to lack of facility with the next two multiples of 4,heturned to using the left hand four counting accrual of 1s while keeping track of the composite units with fingers on the right hand. Indeed, Jake's solutions to this more challenging (for him) task differed, and folded back, from his coordinated actions to solve the previous tasks (with unit rates of 5 and 2). This suggested that Jake's goal-directed coordination of double counting operations on composite units was still evolving and thus dependent on the numbers used in the task (in the sense of his facility with the multiples of the unit rate).

To further demonstrate the interplay between numbers chosen for a task and a child's ways of operating, we present his solution to the task that followed (Excerpt 4). Here, the teacher decided to keep the number of towers at 3, but to increase the unit rate (6 cubes per tower) so it exceeds the number of fingers on one hand. This choice intended to explore how Jake would cope with the challenge of keeping track of composite units while using fingers on both hands to keep track of a sequence of multiples with which he was not facile.

Excerpt 4

Nina: How many cubes did you bring altogether?

Jake: (Silently touches each cube in the first tower, perhaps re-counting them all. He then holds up and stares at his two hands for 1 second, puts his hands down, reaches for a pen, looks back at the cubes, points to the cubes for 1 second, and writes on a small white board the number 6 first and then the letter "T" above it.)

T
6

He looks back at the towers on the table and writes on his white board another set:

T	T
6	6

He looks back a third time at the towers, and completes his writing to correspond with the numbers given in the task:

TTT		
6	6	6

Seemingly not knowing how to proceed, he turns back to the real towers; taps on each of 8 cubes individually (all cubes in the first tower and two from the second), stops, and erases the white board completely. At this point, he begins counting while pointing with his finger to each individual cube in the first tower, shifting to counting by 2s to expedite the process, finally saying): I brought 18 cubes altogether.

Nina: (Not indicating if he was correct or not) How did you figure it out?

Jake: (Picks up all of the towers and then puts them back on the table) I figured it out a different way; I just counted it this time. I used the cubes (picks up one tower). I pretended I broke it up like … (breaks off individual cubes from each tower); like 1-2-3-4-5-6;that is a tower I counted.

Nina (a little later): Did it get a bit harder today when the numbers got harder?

Jake: It was hard for me because I wanted to use the way I did last time (holds up both hands and shows the hand motions that he made when using his fingers to re-present a coordinate, double-count with 4s); but I don't have as much fingers.

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Nina: So you couldn't, you couldn't, you don't have as much fingers so does that mean you couldn't? What do you mean by that (shows holding up of both hands)?
Jake: I had to do it a different way.
Nina: What do you mean you don't have as much fingers?
Jake: I only have 10 fingers and … (rebuilds the 3 towers of 6 using the cubes on the table) Since …So if there are towers of 6 (holds up both hands) and I went like that (begins to count out six on his right hand);So I wouldn't have enough fingers.

Excerpt 4 provided further evidence how the numbers Jake operates on make a difference in his goal-directed activity. When presented with a unit rate of 6, for which he had neither a mental number sequence nor enough fingers to re-present the items, he seemed unable to complete a coordinated count the way he did for unit rates of 2 or 5. Jake's first, spontaneous attempt to re-present the towers was to write a "T"(for tower) and the number"6" under each "T" to indicate how many 1s constituted that composite unit. While insightful and resourceful, this symbolized re-presentation did not proceed to a double-counting activity. Seemingly having no other recourse, Jake abandoned this initiative (e.g., erasing the white board) and instead folded back to counting each of the tangible cubes (albeit shifting from counting by 1s to counting by 2s to expedite the process). Jake's explicit utterance, about not having enough fingers, indicated he was acutely aware of the need to but unable to operate in a coordinated way when the number in one, unfamiliar composite unit precluded using each hand for a different component of the coordinated count. This was evident in his show of two hands, used to this end for 3 towers of only 4 cubes each, and the statements that followed ("I had to do it a different way"; I only have 10 fingers").

Discussion

This paper focused on a key consideration for teaching specific mathematical ideas to students, namely, the choice of numbers used in tasks. Particularly, our study focused on this consideration when using tasks to foster students' construction of the multiplicative double counting (mDC) scheme. Our study can contribute to the field in two important ways. First, it provides further support to the stance that construction of a particular scheme (e.g., mDC) is not a "once-and-for-all" event. Rather, when a shift in the child's way of operating is to be promoted, such as the cognitive leap from additive to multiplicative reasoning (Steffe, 1992), initial emphasis in task design and implementation needs to be placed on orienting the child's mental powers onto the novel coordination of operations on units. To this end, choosing 'easy' numbers seems highly productive, because the child can bring forth available knowledge of numerical calculations with which she or he seems facile (e.g., multiples of 2s and 5s). Initially solving tasks that involve 'easy numbers' enables the child to construct the intended goal-directed activities (a new scheme), which can then become an invariant way of operating she or he could apply to solving tasks with numbers that require engaging more complex mental capacities. (The scope of this paper did not allow us to provide data on how that change was successfully fostered in Jake after what we have seen in Excerpt 4.)

Second, our findings provide further support to the core construct of folding-back in Pirie & Kieren's (1994) model of growth in understanding. Specifically, we showed how Jake was able to use his newly constructed understanding of the coordinated count of composite units and begin operating in multiplicative situations when using 'easy numbers' (Excerpts 1 & 2). Nevertheless, he folded back to operating on 1s, while also conflating unit rate with the number of composite units (Excerpt 3), and was initially unable to solve similar (from an adult's point of view) tasks presented with numbers for which he had no facility with the multiples. He further folded back, foregoing use of double counting when the unit rate (6) exceeded what he could signify with one hand's fingers. As Tzur and Simon (2004) noted, folding-back may be a good behavioral indication for a stage in

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constructing a new scheme at which the child's evolving scheme (here, multiplicative coordination/distribution of composite units) is yet to become independent and spontaneous. This seemed to be Jake's case, when anticipating he would not have enough fingers to account for all 1s in a tower of 6 cubes, and thus producing the abstract diagram to create a tool for keeping track that seemed not yet readily available for him conceptually. Folding-back can thus also be a good indicator of the need to pay close attention to the child's conceptualization, in the following, threefold sense: (a) not simply attributing to the child too high a level of conceptual growth due to successful performance on 'easy-number' tasks; (b) not attributing to the child outright failure to conceptualize the intended math due to unsuccessful performance on 'harder-number' tasks; and (c) designing tasks and using tools that can gradually bring the intended mathematics to be within the child's mental reach.

Endnote

[1]To improve readability, the authors have elected to use numerals in the transcriptions when speakers make number references.

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