STUDENT UNDERSTANDING OF DIRECTIONAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

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Action-Process-Object-Schema (APOS) Theory is applied to study student understanding of directional derivatives of functions of two variables. A conjecture of the main mental constructions that students may do in order to come to understand directional derivatives is proposed and is tested by conducting semi-structured interviews with 26 students who had just taken multivariable calculus. The interviews explored the specific constructions of the genetic decomposition that student are able to do and also the ones they have difficulty doing. The conjecture, called a genetic decomposition, is largely based on the elementary notion of slope and on a development of the concept of tangent plane. The results of the empirical study suggest the importance of constructing coordinations of plane, tangent plane, and vertical change processes in order for students to conceptually understand directional derivatives.

Keywords: Advanced Mathematical Thinking

Introduction and purpose of the study

The calculus of functions of several variables is of fundamental importance in the study of mathematics, science, and engineering. Some work has been published regarding functions of two variables (see for example, Trigueros and Martínez-Planell, 2010; Martínez-Planell and Trigueros, 2012). However, there are very few publications on the differential calculus of such functions. The only publication we'll refer to, Weber (2012), includes a discussion of the rate of change of functions of two variables focusing on the use of covariational thinking to help students build a notion of rate of change in space which centers on students' development of a symbolic representation of the directional derivative. In this paper we focus on students' geometrical understanding of directional derivatives and its relationship to other important ideas in the schema of the differential calculus of functions of two variables. Our research questions are: What are students' conceptions of directional derivatives after taking a Multivariate Calculus course? What are the main mental constructions involved in learning this concept?

Theoretical framework

APOS Theory is used as a theoretical framework to study the cognitive development of students who completed a course using a traditional lecture/recitation model, as discussed in Arnon et al. (2013, p. 106). As APOS is a well-known theory it is briefly discussed (see Figure 1). In APOS, an Action is a transformation of a mathematical object that is perceived by the individual as external. It could be the step by step implementation of an algorithm according to explicit instructions or the application of a fact or result that has only been memorized. Activities that lead students to repeat and reflect on an action can help them to interiorize the Action into a Process. A Process is characterized by the individual's ability to imagine doing the main Actions and to anticipate their result without having to explicitly perform them; In a Process the Actions are perceived as internal. A Process may be coordinated with other Processes, and may also be reversed. As an individual needs to apply Actions on a Process, he/she may become aware of the process as a totality. In this case,

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when the individual applies or can imagine applying Actions to the Process, then it is said that the Process has been encapsulated into an Object. Actions, Processes, and Objects may be organized into Schemas. A Schema for a given mathematical notion is a coherent collection of Actions, Processes, Objects, and other Schemas that are related in the individual's mind to the notion that is being considered. Actions on a Schema may result in its being thematized into an Object. Schemas develop as relations between new and previous Actions, Processes, Objects and other Schemas are constructed and reconstructed. Their development may be described by the Intra-, Inter-, Trans- "triad": At the Intra- stage relations among the Schema components are being constructed but they remain for the most part isolated from one another. At the Inter-stage, transformations between some of the Schema components are recognized. The Trans- stage is defined in terms of the construction of a synthesis between them, so that the Schema is coherent and the individual can decide when its use in problem solving is needed.

Also, although it might be thought that in APOS theory there is a linear progression from Action to Process to Object and then to having different Actions, Processes, and Objects organized in Schemas, this often appears more like a dialectical progression where there can be partial developments, passages and returns from one conception to another. What the theory states is that a student's tendency to deal with problem situations in diverse mathematical tasks involving a particular mathematical concept is different depending on whether the student understands the concept as an Action, a Process, or an Object or has constructed a coherent Schema.

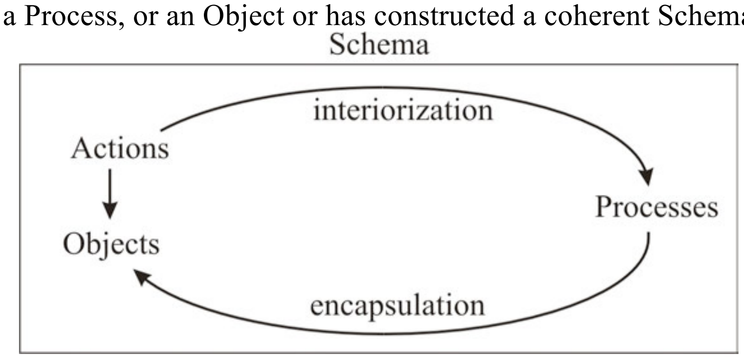

Figure 1: Mental structures and mechanisms

The application of APOS theory to describe particular constructions by students requires that researchers develop a genetic decomposition - a model that describes the specific mental constructions a student may make in understanding mathematical concepts and their relationships. As a model, a genetic decomposition predicts the constructions needed to learn the concepts of interest. It is proposed by researchers and needs to be tested experimentally. The genetic decomposition that follows was developed from reflection on the mathematics itself, considering what it takes to make the idea of a directional derivative understandable to students, and the classroom experience of the researchers implementing initial versions of the idea for several consecutive academic years.

Our genetic decomposition of the directional derivative is essentially based on the notions of directed slope in R^3 and vertical change on a plane. Moreover, the genetic decomposition of vertical change on a plane can be used as a starting point to describe the mental construction of several important concepts of the differential calculus of functions of two variables including tangent plane, differentials, and directional derivatives. Hence, while based on the elementary idea of slope, it can potentially provide a unifying framework for the description of the main ideas of the differential calculus of two-variable functions, thus contributing to help students construct, at least, an Inter-Schema stage of development for the differential calculus of functions of two variables.

The genetic decomposition is as follows: Given a non-vertical plane, the Processes of slope of a line and fundamental plane (planes of the form $x = c$, $y = c$, $z = c$) are coordinated into new processes of vertical change in the x and y directions, where it is recognized that vertical change in the x direction can be described as a function of the horizontal change in the x direction ($\Delta z_x = m_x \Delta x$), and similarly for vertical change in the y direction ($\Delta z_y = m_y \Delta y$). These processes are coordinated into a Process of total vertical change on a plane, so that total vertical change in any plane is given in terms of the sum of vertical changes in the directions of the coordinate axes: $\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$ (see Figure 2). Actions and Processes which are treatments and conversions in and between representations (Duval, 2006) are performed on the Process of total vertical change to encapsulate it into the Object conception of plane in three dimensions. In particular, the point-slopes formula for a plane, $z - z_0 = m_x(x - x_0) + m_y(y - y_0)$ may be seen as the vertical change from an initial point (x_0, y_0, z_0) to a final generic point (x, y, z) on the plane.

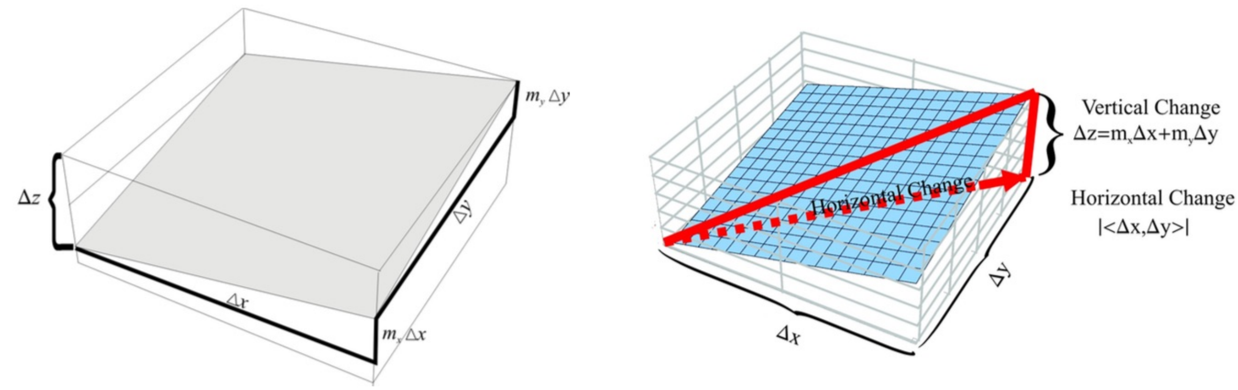

Figure 2: Vertical change on a plane and directional derivative

The Process of partial derivative is coordinated with that of plane in three-dimensional space into a new Process where tangent planes to any surface at different points can be considered and calculated. When there is a need to consider particular tangent planes and perform actions on them to describe the surface in terms of the behavior of partial derivatives, this Process is interiorized into an Object conception of tangent plane.

To do the mental construction of $D_{\vec{v}} f(a,b)$, the directional derivative of f in the direction $\vec{v} = \langle \Delta x, \Delta y \rangle$ (not necessarily unitary) at the point (a,b), the student may coordinate the Process of three-dimensional space with the Process of function of two variables in order to locate and represent in space or imagine the point $(a, b, f(a,b))$. Further coordination with the Process of vectors allows use of the point $(a,b,0)$, or more generally any point of the form (a,b,z) as a starting point from which to represent the direction vector $\vec{v} = \langle \Delta x, \Delta y \rangle$ in space as $\langle \Delta x, \Delta y, 0 \rangle$. Then, the Processes of vector, slope, and derivative of function of one variable are coordinated to represent physically or geometrically, and recognize, the directional derivative as the slope of the line tangent to the graph of the function at the point $(a, b, f(a,b))$ in the given vector direction (a directed slope). To obtain the value of $D_{\vec{v}} f(a,b)$ (see Figure 2), the student would then coordinate the Process of slope of a line with the Process of tangent plane to obtain the vertical change as $f_x(a,b)\Delta x + f_y(a,b)\Delta y$, and with a Process of vector magnitude to obtain the horizontal change as the magnitude of the direction vector $|\vec{v}| = |\langle \Delta x, \Delta y \rangle| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, and thus obtain $D_{\vec{v}} f(a,b)$, as $\dfrac{f_x(a,b)\Delta x + f_y(a,b)\Delta y}{|\langle \Delta x, \Delta y \rangle|}$. These

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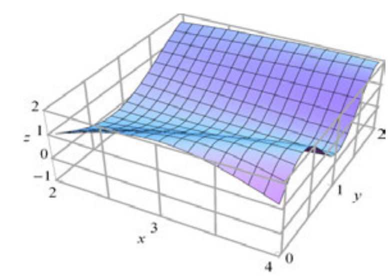
Processes and coordinations must be constructed in different representations. The computation of $D_{\vec{v}} f(a,b)$ at a fixed point (a,b) for different direction vectors $\vec{v}$ and at different points (a,b) for a fixed direction vector $\vec{v}$ allows the encapsulation of the directional derivative Process as a function that depends on the direction vector $\vec{v}$ while also recognizing the functional dependence of the directional derivative on the starting point (a,b).

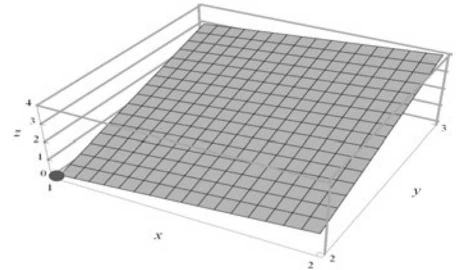
Method

An instrument consisting in 6 multi-task questions was designed to conduct semi-structured interviews with 26 students to test their understanding of the different components of the proposed genetic decomposition. Of these 6 questions we will only report on the 2 which directly considered directional derivatives. All participants were students of science and engineering that had just finished a course on multivariable calculus. Nine (9) of them came from a group where a traditional teaching approach was followed, and the other 17 students came from two groups where activities designed in terms of the genetic decomposition were used. All students used the same textbook (Stewart, 2006) and covered the same material in the course. The three instructors of the groups were asked to choose 9 students from each of them, considered as above average, average, or below average based on their performance in the class, providing as balanced a distribution as possible, to participate in the interviews. One of the students did not show up to the interview and hence we were left with a total of 26 students. All participating professors were experienced (with at least 20 years of experience and having repeatedly taught the course during those years), popular with students (judging on how fast their sections fill up and on student evaluations), and had, throughout the years, shown concern about student's learning. Each interview lasted on the average from 40 minutes to 1 hour. The interviews were recorded and transcribed. Data was independently analyzed by the researchers and conclusions were negotiated among them. We now discuss the questions that dealt directly with the directional derivative and that were analyzed in terms of the structures described in the genetic decomposition:

1. Suppose the graph of $z = f(x,y)$ is as follows. State the sign (positive, negative, zero) of $D_{<-2,1>} f(4,0)$. (This is the second part of a question in the original interview instrument. In the first part, students used the same graph to determine the sign of $f_y(4,0.7)$.)



2. The following plane is tangent to the graph of $z = f(x,y)$ at the point (1,2,0). Use the given figure to find $D_{\langle 1,1\rangle} f(1,2)$.



It is important to note that in the first question, the function initially is increasing in the given direction and thus the directional derivative is positive, $D_{<-2,1>} f(4,0) > 0$, while in the second one, thinking of the directional derivative as a slope, the vertical change may be obtained looking at the given figure as $\Delta z = 4 - 0$, and the horizontal change as $|\langle 1,1\rangle| = \sqrt{2}$. Hence the value of the directional derivative is $4/\sqrt{2}$. Using the Process of vertical change on a plane this may also be

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obtained as $D_{\langle 1,1\rangle} f\left(1,2\right) = \dfrac{m_x \Delta x + m_y \Delta y}{\left|\langle 1,1\rangle\right|} = \dfrac{1(1)+3(1)}{\sqrt{2}} = \dfrac{4}{\sqrt{2}}$.

Results

It seems to be commonly assumed in instruction that students can readily represent the vector direction in a directional derivative. However, we found that frequently this is not the case. Tania, one of the best performing students, could quickly and without hesitation identify the sign of $f_y\left(4,0.7\right)$ in problem 1, but seemed unable to represent the vector direction and thus was not able to give the sign of $D_{\langle -2,1\rangle} f\left(4,0\right)$.

Interviewer: Will that directional derivative be the slope of a tangent line or not?
Tania: Yes, it is the slope of a tangent line.
Interviewer: Of what line?
Tania: That's the tricky part. That's the line I'm looking for.

In the case of David it was observed that in problem 1, he made the necessary coordinations to identify the base point and correctly represented the direction vector. However, he did not coordinate the vector and derivative of a function of one variable Processes. Furthermore, he remembered a formula that would give him a correct answer but did not seem to have constructed a Process of vertical change on a plane that would enable him to give geometric meaning to it:

David: The directional derivative at the point $\left(4,0\right)$ … x is -2, y is 1, it goes this way…

[Correctly representing the vector direction with a dashed line starting at the base point; see Figure 3.] Then the directional derivative at this point will be equal to …
$\dfrac{f_x(4,0)(-2)+f_y(4,0)(1)}{\sqrt{5}}$ … [He went on to interpret this as the slope of the secant line he drew in Figure 3.]

After a while, the interviewer suggested to David that he think of a tangent line. He then managed to correctly draw a tangent line (see Figure 3), however, the slope he came up with was not the **directed** slope:

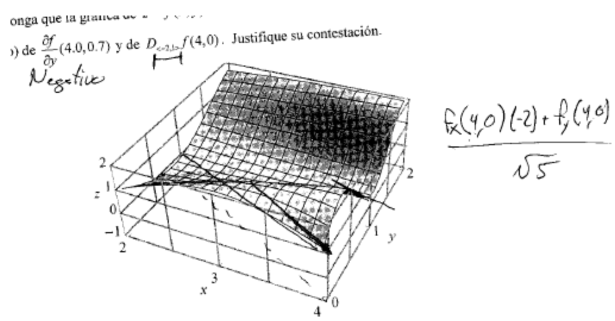

Figure 3: David's drawing on problem 1

Interviewer: And if I were to tell you that the directional derivative is the slope of a tangent line, could you draw the tangent line to that graph at that point in that direction?
David: At this point in that direction? [He draws a tangent line that seems to be correct.]
Interviewer: Will that slope be positive or negative?
David: it will be a negative slope… it is negative because… while the value of z decreases, the value of x increases, so it would, it would be negative, it would be coming down.

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Even though David manages, without hesitation, to obtain the correct answer to problem 2, he gives no evidence of having coordinated the Processes of vertical change and plane or of tangent plane to give geometric meaning to his conception of directional derivative:

David: Use the given figure to find … it would be the partial with respect to x at the point (1,2) times 1, the partial with respect to y at the point (1,2) times 1, over the square root of 2… 4 over the square root of 2 (see Figure 4).

Overall we can only say that David seems to show a conception of directional derivative that is in transition from Action to Process because, even though he showed some evidence of having constructed a Process conception of directional derivative, as evidenced by his ability to locate base points, represent vectors' directions, and do some computations, he was still mostly dependent on a memorized formula.

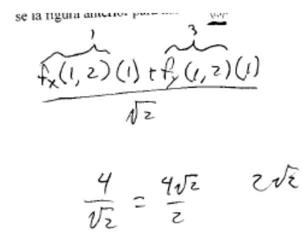
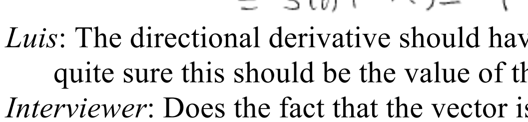


Figure 4: David's written answer to problem 2

Some students like Luis, were able to do these coordinations to explain correctly the expected answer. His behavior on problem 1 is consistent with a Process conception of directional derivative. Of course, his behavior on just one problem is not sufficient to guarantee he has constructed this type of conception of directional derivative, since a student's conception can only be ascertained by considering the student's **tendency** to deal with different problem situations involving directional derivatives.

Luis: [On problem 1] The directional derivative, I'm given (4, 0). Here I have located the point. I have to look for the direction $\langle -2,1 \rangle$ which would be 2 units to the left on the graph, that vector, and 1 unit to the right in y. It should be positive there since the slope of z [he probably means the value of z] in that direction is increasing.

Continuing with the genetic decomposition, after the student was able to locate the base point and the vector direction, and after coordinating the Processes of vector and that of derivative of function of one variable to think of the problem in terms of tangent lines and directed slopes (in problem 1), the student was expected (in problem 2) to coordinate the Process of slope with the Process of tangent plane in order to compute the vertical change along the plane and to coordinate this last Process with that of vectors in order to compute the horizontal change as the magnitude of the direction vector. However, in problem 2, Luis was not able to coordinate the Processes of vertical change, horizontal change, and slope of a line to deal with the directional derivative, but rather seemed to be depending on a memorized and unconnected formula (used in the class textbook; Stewart, 2006) which is valid only for unit direction vectors. After writing:



Luis: The directional derivative should have the value of 4. I'm not completely sure, but I am quite sure this should be the value of the directional derivative.

Interviewer: Does the fact that the vector is not unitary play any role?

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Luis: … Well, in this case, since it is a plane… I believe that in this case since it is a plane it won't make much of a difference… the directional derivative will have the same value as long as it is in the direction $\langle 1,1\rangle$ … maybe if it were a more complex graph… in this case it shouldn't be. Maybe there's a problem but I'm not sure.

Although Luis showed that he was certainly on his way to constructing a Process understanding of directional derivative, he did not show he was able to coordinate the Process of vertical change on a plane needed to interpret $D_{\langle \Delta x,\Delta y\rangle} f(a,b)$ as a slope with vertical change given by $f_x(a,b)\Delta x + f_y(a,b)\Delta y$. The reason for this is, probably, that in his classroom and in the textbook, $D_{\vec{u}} f(a,b)$ was defined only for a unit direction vector $\vec{u}$ and the geometric interpretation of the directional derivative as a slope remained hidden both in a graph and in the formal development of the mathematics (see Stewart, 2006).

Daylin was able to compute the directional derivative in problem 2 while justifying geometrically her computations. Hence she seems to have constructed the conjectured coordinations needed to be able to think of the directional derivative as a slope and obtain the necessary vertical change and horizontal change (see Figure 5). Further, since she also gave evidence of doing the conjectured mental constructions required to solve and explain problem 1, one may reasonably state that she seems to have constructed a Process conception of directional derivative.

Daylin: Then the height at the new point will be 4, if I put this triangle [see Figure 5] my height here is 4… this is the direction $\langle 1,1\rangle$ and I want the slope… here z is zero, I have the rise, I need the run, the run will be … the square root… so the slope will be 4 over $\sqrt{2}$.

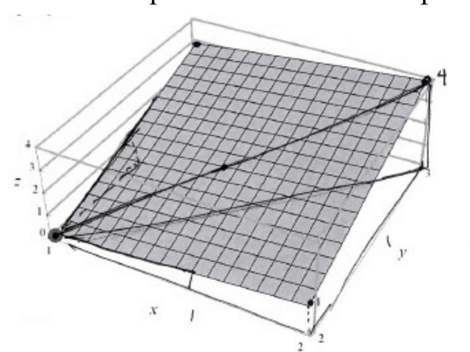

Figure 5: Daylin's drawing on problem 2

Summary and Discussion

Only 4 of 26 students gave evidence of having made all or most of the mental constructions required in the genetic decomposition. This tells us that the idea of a directional derivative is difficult for most students and they need more help understanding even the most elementary notions associated with this idea. It also suggests that much work still remains to be done designing and improving activities to help students do the conjectured constructions.

We just saw that although teachers probably frequently assume that students will be able to imagine the base point and the role of the vector direction to understand directional derivatives, this seems not to be the case for many students, and that explicit attention to the necessary coordinations can help students make the desired mental constructions. Also, many students did not show a Process conception of derivative of function of one variable to be used in the proposed mental constructions of directional derivative. This suggests that rather than assume they have constructed this Process, instruction can start by, once again, explicitly considering ways to help them construct it, but now in the context of functions of two variables. The construction of a Process of vertical change on a plane

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was shown in this study to be important to facilitate the mental construction of the processes of tangent plane and directional derivative that help students give geometrical meaning to directional derivative, and that students may not be able to construct it without explicit attention in their classroom instruction.

Overall, results show that most of the students who had successfully finished a course on multivariable calculus did not have a deep understanding of the concept of directional derivative. Many students relied on memorized facts and formulas and thus showed difficulties when responding to questions that needed a deeper conceptual understanding. Most students lacked geometrical understanding of the basic components involved in the definition of a directional derivative. The lack of understanding of those concepts impaired them from realizing that the symbol $D_{\vec{v}} f(a,b)$ denotes a special type of derivative and that, as such, it can be represented as a slope of a tangent line in the given direction, as well as understanding that, in three-dimensional space, the notion of slope would be ambiguous unless it is a directed slope, and recognizing that for a vector direction $\langle \Delta x, \Delta y \rangle$, which is not necessarily unitary, the directional derivative represents the directed slope of a line on the tangent plane with vertical change given by $f_x(a,b)\Delta x + f_y(a,b)\Delta y$ and horizontal change given by the magnitude of the direction vector $\langle \Delta x, \Delta y \rangle$. This is necessary to make sense of the formula traditionally used in textbooks, where the direction vector $\langle u_1, u_2 \rangle$ is unitary.

The assumption that students will, look at textbook figures and on their own, come to understand the geometric ideas involved in directional derivatives, apparently held by many teachers who just present students with the formula $D_{\langle u_1,u_2 \rangle} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$, seems not to be valid since the simple geometrical explanation of the notion of directional derivative as a slope remains hidden. Of course, any instructional approach will need to eventually consider only unitary direction vectors since this formula can be expressed as the dot product of the gradient vector and the unitary direction vector, an observation which is crucial in exploring the properties of the gradient vector. Results obtained suggest that instruction might start by exploring the basic property of slope where the vertical change is seen as the slope times the horizontal change, $\Delta V = m\Delta H$, and then interpreting this in the case of a plane to obtain the basic idea of vertical change on a plane, symbolically represented by $\Delta z = m_x \Delta x + m_y \Delta y$. This is essentially the idea inspiring the genetic decomposition presented. A possible virtue of this approach is that the notions of plane, tangent plane, the differential, and the vertical change in a directional derivative are all explained and inter-related by this simple geometric idea, thus potentially helping students build a coherent schema for the differential calculus of functions of two variables. But, this remains to be investigated.

Acknowledgements

This project was partially funded by Asociación Mexicana de Cultura A.C.

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