THEORIZATION OF INSTRUMENT DESIGN IN RESEARCH ON MATHEMATICAL PROBLEM SOLVING

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In this paper we report the interview questions used in a study of middle school students' mathematical problem solving behaviors which were chosen based on Vygotsky's concept formation theory and Berger's appropriation theory. We discuss the task design/selection process along with the findings associated with the use of these tasks so to provide new direction for gauging research on mathematical problem solving.

Keywords: Research Methods; Problem Solving; Geometry

Introduction

Improving the teaching and learning of mathematical problem solving relies heavily on development of a theory of mathematical problem solving, which is currently missing from the field (Schoenfeld, 2007). As Schoenfeld articulated, the focus of mathematics education researchers interested in mathematical cognition may need to shift on building theoretical capacity that may account for human decision making in the course of problem solving (2013). We posit that progress towards building such a theory might demand greater attention to theorizing instrument design and task development in conducting research on mathematical problem solving, an area currently absent from the field. Discussions surrounding tasks have frequently focused on defining problems: a question where an individual does not have a ready- to-use approach to find the answer (Wilson, Fernandez, & Hadaway, 1993). Agreement exists that whether a question is a problem depends on the individual working on the task (Schoenfeld, 1985). This description has imposed constraints on researchers' ability to theorize specific principles of instrument design/selection when undertaking research on mathematical problem solving. Existing studies have generally selected questions based on the targeted subject area and whether the question is appropriate for the participants in the study, referring to the participants' educational background as a standard to determine whether a question is beyond their capability or not (e.g. Elia et al., 2009; Kuzle, 2011). These efforts, although useful in providing a profile of expert problem solvers, do not provide a coherent perspective on instrumentation as a methodology.

In a larger study of middle school students' mathematical problem solving behaviors, we used research-based assessment to capture students' ways of knowing by unpacking the relationship among mathematical concepts, cognitive behaviors, and metacognitive behaviors as evidenced during clinical interviews. We reported the findings from the pretest instrument in the study, which suggested that Vygotsky's concept formation theory could serve as an effective framework for designing novel assessments to provide researchers with more precise tools to articulate intricacies of students' understanding of mathematical concepts (Zhang, Manouchehri, & Tague, 2013). In this paper we will report on theoretically based criteria for design of tasks to be used in our research on mathematical problem solving. We will report findings related middle school students' performance on a selected sample of these tasks so to provide direction for future research on mathematical problem solving.

Theoretical framework

As a starting point in our task design, two issues were of particular concern: (1) establishing theoretical capacity that would allow us to document and analyze both cognitive and metacognitive

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behaviors of the participants in the course of their problem solving; (2) selecting a specific mathematical concept so to develop appropriate tasks surrounding it. To address the first issue we incorporated two theoretical perspectives to design/select interview questions used in this study: Vygotsky's (1962) concept formation theory and Berger's (2004) appropriation theory. Vygotsky's theory proposes a framework for an individual's concept development within a social environment, while Berger's theory proposes an interpretation of Vygotsky's theory in the domain of mathematics by adjusting certain stages. Both theories break down any concept development into three phases: heap, complex, and concept. In the heap phase, the learner associates a sign with another because of physical context or circumstance instead of any inherent or mathematical property of the signs. In the complex phase, objects are united in an individual's mind not only by his or her impressions, but also by concrete and factual bonds between them. In the concept phase, the bonds between objects are abstract and logical. For the purpose of the study we also decided to focus on the concept of Area due to its critical role in school mathematics. Hence, research-based formation stages for the concept of area were assembled as illustrated in Figure 1.

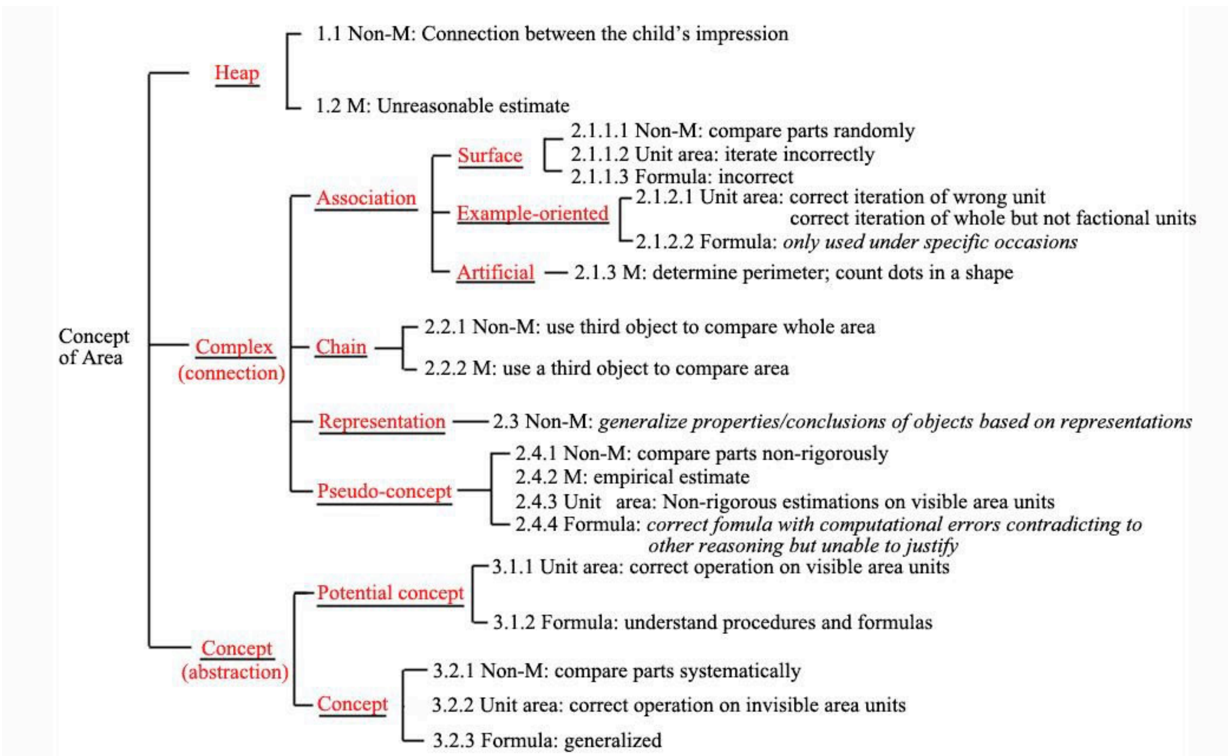

Figure 1. Developmental Stages of the Concept of Area

This structural model served as our primary analytical tool for qualifying the students' work as well as the tasks selected to be used in study. This model was further refined upon a short pilot study in which responses to tasks were obtained from 44 middle school students. The students were asked to responds to five items. Upon analysis of their responses we considered revisions of some of the tasks so to assure ambiguities that could lead to irrelevant answers were removed. Analysis of the relationship among mathematical concepts, cognitive and metacognitive behaviors emerged from the interview results based on our final selection of items which were used in in-depth interviews with five middle school students. The current report is based on our findings of these interviews.

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Methods

Participants

Five individuals from a population of 44 sixth grade students were selected to participate in interviews. The original group of 44 were enrolled in three distinct class periods of an algebra course taught by the same teacher. All three classes had been observed by the lead author for 6 months prior to data collection. A pretest was administered to the 44 students. The process of determining and classifying the pretest responses is described as the following. 1) Two authors independently reviewed all responses to identify and document enacted approaches and coded developmental stages associated with each approach based on an earlier version of the developmental stages framework. Notes were compared for consistency in coding. 2) Students' approaches that were ambiguous or non-anticipated were discussed. The framework was adjusted based on the analysis of these responses; five more stages were identified and added to the original framework. 3) Based on the analysis we built a reference list of detected stages for each student to inform participant selection. Each individual's developmental status revealed in the responses was categorized as "overall low" (all responses were rated as Heap and non-Pseudo- concept Complex stages), "varied" (responses were rated across Heap to Concept stages), and "overall high" (Reponses were rated as Pseudo-concept Complex and Concept stages).

Participant selection was deliberate. The following criteria guided our choices: 1) willingness to be involved in the study and had signed the consent forms; 2) the participants would need to represent a range of different developmental attributes pertaining to the target concept. This data was collected through the students' pretest responses; 3) the participants would need to be comfortable with thinking aloud. Five individuals met these criteria: Shana exhibited a *low* status, Andy exhibited a *high* status, Sandy, Allen, and Ivan exhibited *varied* status.

Data collection

The five participants were interviewed individually. Each interview consisted of two parts: During the *background interview* part the participants' mathematics background information, their beliefs about mathematics, and their views on the value of mathematics for their lives were elicited. The second part, *problem solving session* the participants worked on specific mathematical tasks. During the problem solving sessions, interviewer interventions were limited to eliciting clarifications, explanations, or justifications, when needed.

Instrument design

F ive problems were used during the interviews. All problems were related to the concept of area and allowed the participants to tackle the tasks from different stages of concept development. The problems could potentially cover a wide range of concept stages. Peripheral concepts were integrated in several problems in order to enable some degree of interactions between different concepts (e.g. transformational reasoning and variable), since we believe authentic problem solving should involve more than one concept yet the number of concepts needs to be controlled for a deeper analysis on the behaviors acting upon them. When selecting tasks, concept stages along with the corresponding exemplar approaches that could be elicited by each problem were predicted by the authors. An example of this conceptual mapping is shown in Table 1.

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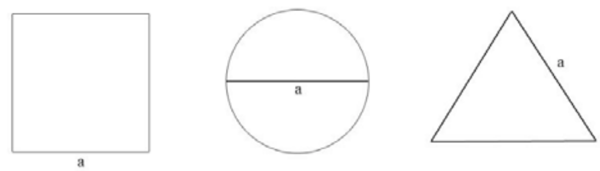
Table 1. Predicted stages and exemplar approaches

Compare areas problem
Which of the regions shown below has the largest area? How would you order them?



Predicted stages	Exemplar approach
Surface Association Complex – Formula	Using an incorrect area formula
Chain Complex	Fitting the circle and the triangle into the square
Potential Concept - Formula	Using a correct area formula

Data analysis

Data collection consisted of four phases. First, each participant's key cognitive behaviors during each problem solving episode were mapped and documents. Second, a summary of observed concept stages and metacognitive behaviors during the episode were catalogued and noted. Detailed analysis of the individual's problem solving behaviors according to the relationship among concepts, cognitive behaviors, and metacognitive behaviors was constructed. Finally, cross analysis of the observed concept stages, metacognitive behaviors, and the relationship between them concluded the analysis phase. This process was followed for each of the five tasks used.

Results

The larger study analyzed the participants' performances on each interview question from four perspectives: 1) each participant's point of entry, including identified task elements/objectives and his/her initial approach, 2) types of approaches the study participants used, 3) concept stages revealed when working on the problem, and 4) metacognitive behaviors revealed from the participants during the problem solving episode.

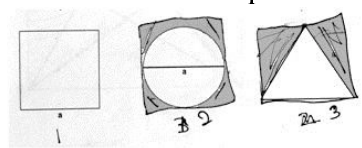
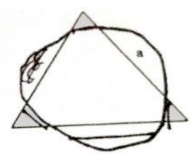
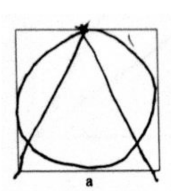
To focus on the research question in this paper, we only report on: 1) each participant's point of entry, including identified task elements/objectives and his/her initial approach, 2) types of approaches the participants used, and 3) concept stages revealed. The problem in Table 1, Compare Areas problem serves as an illustrative example of research-based tasks used in the larger study reported elsewhere (Zhang, 2014).

Point of entry

In the Compare Areas problem, one task element and one task objective are essential to solving this problem: the variable a that represents the equal length of sides and diameter, and the comparison of the areas, respectively.

Table 2 summarizes the task elements/objectives (i.e. key conditions and goals to solve the question) identified by each participant and by the interviewer before each of the participants started solving the problem. Initial approaches adopted by each participant are also outlined.

Table 2. Summarization of initial response/approach adopted by each participant

Participant	Task elements or objectives identified by the participant	Task elements or objectives highlighted by the interviewer	Initial approach
Shana	Equal measure of sides and diameter	NA	The circle and the triangle could fit into the square. 
Sandy	Just area	The *a* meant equal measure of sides and diameter	Computed the areas by formulas.
Ivan	NA	NA	The circle could fit into the square (verbal conclusion); the triangle had the same area as the circle since the three vertices of the triangle "pokes out." 
Andy	Around is the circumference.	The *a* meant equal measure of sides and diameter	Computed the areas of the square and the triangle by formulas.
Allen	The area inside	NA	The circle and the triangle could fit into the square. 

Among the five participants, Sandy's understanding of variable greatly influenced her initial behaviors. She focused on the actual amount of area instead of the condition that the lengths of the sides and the diameter had the same measures. This prevented her from adopting either a visual or a numerical approach to start tackling the problem. Allen, whose understanding of variable was restricted, chose to ignore it so to avoid confusion. Ivan overlooked the variable; his later behaviors

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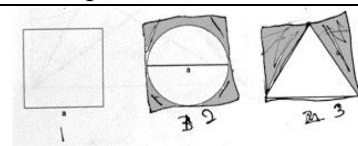
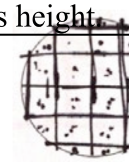
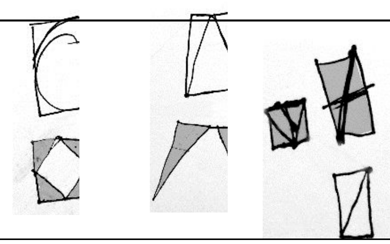
were influenced by this element rather than at the beginning. Interestingly, Shana (who was assessed as low concept development) was the only participant who explicitly identified the task element of equal measure, whereas the others were either confused by the information, overlooking it, or needed to be reminded by the interviewer.

Documented approaches

Table 3 illustrates the approaches that the participants used when solving the Compare Areas problem.

Table 3. Approaches adopted by the participants solving the Compare Areas problem

	Approach	Description	Example from interview
1	Fit into the square	Showed the circle and the triangle could fit into the square.	
2	Formula	Used area formulas to compute the areas.	Assigned a number to *a* and used the measure of the side of the triangle as its height.
3	Unit squares	Drew and count unit squares to approximate the area.	
4	Compare leftover	Transformed the leftovers of the circle and the triangle into manageable shapes for comparison.	

The third approach was adopted when a specific area formula (i.e. the area formula of circles) was not available to Andy who used the second approach as his initial heuristic. The fourth approach was usually adopted when the participants, who used the fit-into-the-square approach as their initial approach, were prompted to further compare the areas of circle and triangle.

Revealed concept stages

When designing the interview instrument, the researchers had predicted six concept stages that could be revealed in this problem: 1) *Surface Association Complex – Formula* wherein a student uses an incorrect formula, 2) *Artificial Association Complex* where perimeter is compared instead of area, 3) *Chain Complex - Non-measurement* wherein the circle and the triangle can fit into the square, 4) *Pseudo-concept Complex - Formula* where visual reasoning contradicts to computational answers, 5) *Potential Concept - Formula* where a number is plugged into a formula, and 6) *Concept – Formula* where variable *a* is plugged into the formula to reach a generalized answer. Among these stages, 1, 3, and 5 were revealed during the interviews, while 2 was observed during the identification of task objectives prior to one participant's initial approach.

Table 4 summarizes the concept stages of Area concept revealed from the five participants when solving this problem. The approaches associated with each stage as used by them are noted as well. The stages listed in italic are the ones predicted to be revealed by the researcher.

Table 4. Concept stages of area and associated approach revealed by participants

	Revealed concept stage	Associated approach
	Surface Association Complex – Non-measurement	Compare leftover
	Surface Association Complex – Formula	Formula
	Chain Complex	Fit into the square
	Pseudo-concept Complex - Non-measurement	Compare leftover
	Potential Concept – Unit area	Unit squares
	Potential Concept - Formula	Formula
	Concept - Non-measurement	Compare leftover

All other stages were observed when the participants sought an alternative way to either refine or complement their initial approaches. The participants' problem solving behaviors were much more novel during this process (thus less predictable) than their behaviors during the initial attempt (i.e. using fit-into-the-square or formula approach).

Discussion

The study results suggested that metacognitive behaviors of the participants are closely linked to their perceived level of complexity of the task and whether they found it appropriately challenging. That was the case regardless of the participants' level of development pertaining to the concept under study (in our case, area, for instance) or their personal preferences for particular strategies. That is, when a task was perceived as too challenging, it was treated as enigma. This limited our capacity as researchers to access cognitive or metacognitive behaviors of the participants. The same occurred when the problem was perceived as trivial. Because of this, we argue that description of what constitutes a problem (in the context or problem solving) needs to be more precisely defined. The common description "whether a task is a problem depends on the individual" (Schoenfeld, 1985) too broad to inform research instrumentation when studying mathematical problem solving. We posit that benchmarks for selection/design/ development of appropriate problems, according to the concept formation stages specific to those embedded in the task, can enable researchers to more adequately elicit student problem solving. This issue is particularly critical if inferences are to drawn regarding the individuals' understanding of a concept in presence of their mathematical problem solving performance.

Since existing literature does not provide a guide that can inform task design/selection for research purposes, focused scholarly efforts towards construction of such a theoretical platform are needed. Towards that goal several key issues merit elaboration, among many include: 1) task elements and objectives that are essential to understanding the problem in light of learning progression, 2) specific concept stages as entry points to the tasks, 3) desired concept stages embedded in solutions, and 4) potential shifts/paths between concept and its associated concept stages.

Discussion of task elements need to identify conditions under which a problem may suit different populations according to their experiences with the concepts involved. Naturally, if the problem solver has had limited or no experience with a particular concept their interpretation of the task or what is expected as an appropriate answer may not match those of the researchers. Discussion of specific concept stages as entry points granted by the tasks allows the researcher to use the same task with different populations who may not share the same concept developmental status. Such an elaboration allows us to determine whether the task might be too challenging/impossible to solve given a specific concept developmental stage. A description of potential shifts and paths can serve as an aid for the interviewers/researchers to gauge their interventions during the interviews (potential prompts/probing

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questions when observing problem solving process). These descriptions mainly determine the capacity of a task. If a task allows only one entry point and one desired stage for the solution, it mostly assesses whether an individual knows the procedure or not, which is commonly used in proficiency tests. The development a detailed theoretical model of task design is a necessary tool towards development of a theory of mathematical problem solving.

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