CATEGORIZING STATEMENTS OF THE MULTIPLICATION PRINCIPLE

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The multiplication principle is a fundamental principle in enumerative combinatorics. It underpins many of the counting formulas students learn, and it provides much-needed justification for why counting works as it does. However, given its importance, the way in which it is presented in textbooks is surprisingly varied. In this paper, we document this variation by presenting a categorization of statement types we found in a textbook analysis. We also highlight mathematical and pedagogical implications of the categorization.

Keywords: Advanced Mathematical Thinking; Curriculum Analysis; Post-Secondary Education

Introduction and Motivation

Consider the following three statements of the multiplication principle[1](MP), seen in Figures 1, 2, and 3. Given that these statements are all meant to describe the same fundamental issue in counting, a number of questions naturally arise. Does Mazur's statement include anything that Roberts and Tesman's does not? Is Bona's set-theoretic statement equivalent to the others? If so, what are pedagogical consequences of such variation? These questions serve as motivation for better understanding how the MP is presented in the current generation of textbooks and what implications such varying formulations might respectively entail. In this paper, we report on a textbook analysis in which we examined statements of the MP, providing a categorization of statement types intended to illuminate mathematical and pedagogical issues related to the MP.

> ***Product Rule:*** *If something can happen in n_1 ways,* ***and*** *no matter how the first thing happens, a second thing can happen in n_2 ways,* ***and*** *no matter how the first two things happen, a third thing can happen in n_1 ways,* ***and*** *…, then all the things together can happen in $n_1 \times n_2 \times n_3 \times \ldots$ ways.*

Figure 1: Roberts & Tesman's (2003) statement of the MP

> ***The Product Principle:*** *In counting k-lists of the form $(l_1, l_2, \ldots, l_k)$, if*
> 1. *there are c_1 ways to specify element l_1 of the list, and each such specification ultimately leads to a different k-list; and*
> 2. *for every other list element l_i, there are c_i ways to specify that element no matter the specification of the previous elements $l_1, \ldots, l_{i-1}$, and that each such specification of li ultimately leads to a different k-list,*
>
> *then there are $c_1 c_2 \ldots c_k$ such lists.*

Figure 2: Mazur's (2009) statement of the MP

> ***Generalized Product Principle:*** *Let $X_1, X_2, \ldots, X_k$ be finite sets. Then the number of k-tuples $(x_1, x_2, \ldots, x_k)$ satisfying $x_i \in X_i$ is $|X_1| \times |X_2| \times \ldots \times |X_k|$.*

Figure 3:Bona's (2007) statement of the MP

TheMP is a fundamental aspect of combinatorial enumeration. It is generally considered to be foundational to many of the major counting formulas students learn and is called by some "The Fundamental Principle of Counting" (e.g., Richmond & Richmond, 2009). Mazur (2009) notes that the MP is "quite flexible and perhaps the most widely used basic rule in combinatorics" (p. 5). Even more, the MP can provide a much-needed source of justification for why many common counting

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formulas work as they do. For key concepts in other domains (such as limit, derivative, the fundamental theorem of arithmetic, etc.), there tend to be clear, agreed upon, consistent definitions provided in textbooks. However, we have found in our experiencethat textbooks vary widely in how they present the MP. Given the importance and the prevalence of the principle, and given the apparent lack of consistency with which it is presented, we decided to study how the MP is treated in a sample of postsecondary Combinatorics, Discrete Mathematics, and Finite Mathematics textbooks. We answer the following research questions:

1. How is thestatement of the multiplication principle presented in postsecondary Combinatorics, Discrete Mathematics, and Finite Mathematics textbooks?
2. What mathematical issues arise in comparing and contrasting different statements of the multiplication principle?

Literature Review

Counting Problems are Important but are Difficult to Solve

Counting problems foster rich mathematical thinking, and they have a number of important applications. However, correctly solving counting problems is challenging, and there are many studies that report on students' difficulties with counting (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004; Hadar & Hadass, 1981). Brualdi (2004) says, "The solutions of combinatorial problems often require *ad hoc* arguments sometimes coupled with use of general theory. One cannot always fall back onto application of formulas or known results" (p. 3). Within the last couple of decades, a number of researchers have investigated reasons for students' difficulties and have made progress toward better understanding students' combinatorial reasoning and activity (e.g., Eizenberg & Zaslavsky, 2004; English, 1991; 1993; Maher, Powell, & Uptegrove, 2011; Tillema, 2013). In spite of such work, however, student difficulties with counting persist.

There is a growing body of research suggesting that students may benefit from explicitly thinking about the outcomes they are trying to count. Lockwood (2014) has proposed a set-oriented perspective toward counting, which entails viewing the activity of solving counting problems as inherently involving structuring and enumerating a set of outcomes. The work herein contributes to current literature that frames sets of outcomes as an indispensable aspect of students' counting. In addition, previous work (Lockwood, Swinyard, & Caughman, 2015) has demonstrated the importance of the MP in counting, and the lack of a well-developed understanding of the MP appeared to be a significant problem and hurdle for the students. It is important to note that the MP as a principle of counting is different than the operation of multiplication. We have found in our experience that students can easily assume that they completely understand the MP in counting because multiplication is a familiar operation for them. As a result, they use the operation frequently but without careful analysis, and they tend not to realize when simple applications of multiplication are problematic. We are concerned by the lack of attention students give to the MP, and we argue that the MP is worthy of further investigation. While some researchers have discussed multiplication within combinatorial contexts (Tillema, 2013), there have not yet been studies that explicitly target the MP, and more attention must be paid to the role of multiplication *in counting*.

Textbook Analyses as Insight into How Concepts are Presented

According to Thompson, Senk, and Johnson (2012), "Begle (1973) found that the textbook is 'the only variable that on the one hand we can manipulate and on the other hand does affect student learning' (p. 209)" (p. 254). Thompson et al., go on to point out that textbooks "help teachers identify content to be taught, instructional strategies appropriate for a particular age level, and possible assignments to be made for reinforcing classroom activities" (p. 254). In light of this, a number of researchers have examined textbooks in order to get a better sense of how ideas are presented to students (e.g., Mesa, 2004). At the post-secondary level, this has been seen in the domain of linear algebra (Cook & Stewart, 2014; Harel, 1987), trigonometry (Mesa & Goldstein, 2014), and abstract algebra (Capaldi, 2013). We follow such researchers in using textbooks to gain insight into how mathematical ideas are presented. A potential limitation of this study is that we are simply looking at textbooks, and we cannot make claims about how ideas in textbooks are actually taught to students by an instructor or are understood by students. Nonetheless, a textbook analysis provides an efficient snapshot of how experts in the field of combinatorics define and frame a foundational concept like the MP.

Theoretical Perspective

Structural vs. Operational Conceptions

In Sfard's (1991) presentation of the dual nature of mathematical conceptions, she highlights a relationship between structural and operational conceptions. This dual nature is reflected in the idea that mathematical conceptions can, on the one hand, be considered as *objects* (a structural conception), but that those same conceptions might also be able to be thought of as *processes* (an operational conception). It is interesting that, in her original descriptions of these ideas, Sfard mentions an analysis of textbook definitions:

> The careful analysis of textbook definitions will show that treating mathematical notions as if they referred to some abstract *objects* is often not the only possibility. Although this kind of conception, which from now on will be called *structural*, seems to prevail in the modern mathematics, there are accepted mathematical definitions which reveal quite a different approach. (p. 4, emphasis in original)

Sfard goes on to say that "The latter type of description speaks about *processes, algorithms, and actions* rather than being about objects. We shall say therefore, that it reflects an *operational conception* of a notion" (p. 4, emphasis in original). Sfard (1991) also emphasizes the complementary relationship between the structural and operational conceptions, noting that, "the ability of seeing a function or a number both as a process and as an object is indispensable for a deep understanding of mathematics, whatever the definition of 'understanding' is" (p. 5). This suggests that there could be benefits to having both structural and operational notions of a concept like the MP, something we address in our results and discussion.

Methods

In order to create a broad list of textbooks that were used in postsecondary Finite Mathematics, Discrete Mathematics, and Combinatorics courses, we compiled a list of universities in the union the top 25 ranked universities, the top 25 ranked graduate mathematics programs, the top 10 ranked liberal arts colleges, and the universities with the 10 largest undergraduate populations (National Universities Rankings, n.d., Math, n.d., and National Liberal Arts Colleges Rankings, n.d., respectively). This represented 52 schools in 26 states. We then identified and added to the list the largest university in the remaining 24 states. In total, we surveyed 76 universities representing all 50 states and including some of the top universities in the country.[2]

For each of these 76 universities, we identified courses from the university catalogs and found titles of the required texts from the department's website, the university bookstore, or online course pages. We found textbooks from 70 of these universities. In total, we found three textbooks from one university, two textbooks from 22 universities, and one textbook from 47 universities. We thus identified textbooks for a total of 94 courses at these 70 universities. Multiple universities used many of the textbooks, and so this search yielded a total of 32 textbooks. We also examined relevant

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textbooks within our own personal and university libraries, and this added 32 textbooks not yet on the list. Therefore, in total we had a set of 64 textbooks, which both provides a sense of how students are being exposed to the MP and also gives a relatively comprehensive picture of ways in which the MP is presented in textbooks. Our analysis and results are based on all 64 of these textbooks. Six textbooks did not include a statement of the MP, and some textbooks included multiple statements, and thus we analyzed a total of 73 statements of the MP in these 64 texts.

Analyzing the Textbooks. Once the list of textbooks was compiled, we digitally scanned the sections of each text that introduced the multiplication principle, including any worked examples and *narratives* (the text surrounding the principle, see Thompson, et al., 2012) that accompanied the statement itself. The authors each independently examined the narrative portion of the texts, including the statements of the MP and any worked examples, recording phenomena and developing categories for what we observed. This is in line with Strauss and Corbin's (1998) constant comparative method of qualitative analysis, where our data consist of the textbook sections. Following the creation of codes, for the sake of reliability, each author analyzed the entire set of texts separately, and we then met and discussed all of the codes until consensus was reached. We also addressed our second research question by more deeply examining mathematical properties of the statements via carefully reviewing and discussing the statements.

Results

Due to space, we share only two aspects of our findings. In this section, we first provide a categorization of statements of the MP (which resulted from investigating the extent to which statements themselves reflect structural versus operational conceptions) and report on the frequencies of statement types. Then, we demonstrate the value of this categorization by highlighting a mathematical implication that emerged from an articulation of the statement types.

Structural versus Operational Conceptions Reflected in the Overall Statements of the MP

Drawing heavily on Sfard (1991), we found that the statements of the MP could be categorizedinto three types: *structural statements*, *operational statements*, and *bridge statements*. Broadly, these three statement types differ in terms of what they state the MP is counting. Structural statements characterize the MP as counting objects (without specifying a process to construct those objects), while operational statements characterize the MP as counting ways to complete a process (without specifying the outcomes of that process). Bridge statements provide a link between the two – they frame the MP as counting objects, but they also specify the counting process that would generate the objects. Thus, in order to code the statements, we looked to see how the statement frames what the MP is counting. Table 2 provides the codes, what we took to be criteria for a statement to receive that code, and an example of a textbook whose statement reflects that code.

Table 2: Criteria for Statement Types

Code	Criteria
Structural	The statement characterizes the MP as involving counting objects (such as lists or *k*-tuples)
Operational	The statement characterized the MP as determining the number of ways of completing a counting process
Bridge	The statement simultaneously characterizes the MP as counting objects and specifies a process by which those objects are counted

Structural Statements. To be coded as a structural statement, a statement had to describe counting a set of objects, without any mention of a process that would generate that set. For example, notice that Bona's structural statement (Figure 3) has characterized the MP as a statement about *k*-

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tuples (ordered sequences of length k), which are objects with an inherent structure. The MP describes the total number of k-tuples from k sets, and it is simply expressed by product of cardinalities of k sets. There is no connection made between those k-tuples and a process that would generate them; the statement is simply presented set-theoretically.

Operational Statements. In contrast to Bona's structural statements, the operational statements frame the MP not as counting structural outcomes, but rather as counting ways of completing a process (and a process is clearly articulated in the statement). Roberts and Tesman (2003) provide a statement (Figure 1) that we coded as *operational*, describing the MP in terms performing a task with t successive operations, and the MP provides the number of ways of completing a task. Notice that the nature of what is being counted – the result of the MP is not the number of objects, but rather it is the number of ways of completing a process. Note these two types of statements naturally reflect the duality between structural and operational conceptions that Sfard (1991) proposes. We also note that some textbooks provided both structural and operational in their narratives.

Bridge Statements. A statement like Mazur (Figure 2) on the one hand reflects a structural framing of the MP (the objects being counted are k-lists), but the statement also explicitly describes an operation for how to construct those objects. Such statements, which we call bridge statements, simultaneously both count objects and describe a process by which to count or construct those objects. In Mazur's case, the k-listshe describes are the same object as Bona's (2007) k-tuples. However, unlike Bona, notice that Mazur (Figure 2) describes an operational process that explains how to generate the objects (k-lists) that are being counted – specifically, he describes, "there are c_1 ways to specify element l_1 of the list." The presence of this explicit connection between the structural and operational framings of the statement led us to code this statement by Mazur as a bridge statement.

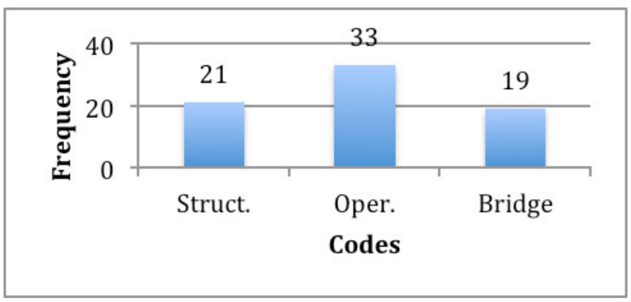

Table 3: Frequencies of structural, operational, and bridge statements (n = 73)

Frequencies. For coding statements at this level, the unit of analysis was a statement of the MP. For any given formulation of a statement, the codes of *structural, operational*, and *bridge* are mutually exclusive, so a statement was coded with exactly one of these codes. Because some textbooks had multiple statements (while some did not include statements), we coded a total of 73 statements across the 64 textbooks. Table 3 shows the respective frequencies of structural, operational, and bridge statements, using the total number of statements as the total frequency.

From Table 3, we observe that operational statements were the most frequent, comprising 45% of the total statements, but each type of statement was represented. These findings convey the variation among statements, supporting the notion that this fundamental counting idea is not presented consistently across textbooks. Through our analysis we also found wide variation in the language used among statements and the representations that accompanied statements, although we do not share those findings here due to space.

Mathematical Implications of Different Statement Types

In this section we address one mathematical implication of different statement types, making a case for what we might gain from a categorization of statement types. As we have noted, the majority

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of the statements in textbooks are operational, framing the MP in terms of counting the number of ways of completing counting processes that have some number of successive stages. A significant issue with these statements is that they make no claim about whether that total number of ways to complete the process are in a one-to-one correspondence with the desirable set of outcomes, and in fact they make no explicit connection to the overall outcomes of the procedure at all. Given our prior focus on sets of outcomes and their importance (e.g., Lockwood, 2013; 2014), the lack of explicit attention to outcomes is concerning.

For example, Roberts and Tesman's (2003) statement (Figure 1) is strictly operational, and we see that the MP yields the number of ways for "all the things together" to happen, but the statement says nothing about the total number of outcomes. We contrast this with Tucker's bridge statement (Figure 4), which describes a process by which to generate outcomes, not the number of ways to complete the process. In fact, Tucker goes so far as to state that, as a condition of implementing the MP, the "distinct composite outcomes must all be distinct."

The Multiplication Principle: *Suppose a procedure can be broken down into m successive (ordered) stages, with* r_1 *different outcomes in the first stage,* r_2 *different outcomes in the second stage, ..., and* r_m *different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has* $r_1 \times r_2 \times \ldots \times r_m$ *different composite outcomes.*

Figure 4 – Tucker's (2002) statement of the MP

In many simple problems, using a strictly operational statement type is not problematic, and any differences between the operational and bridge statements may seem immaterial. For instance, consider the question "*Suppose we flip a coin 10 times in a row. How many possible ways are there to do this?*" Here, we can solve the problem by thinking of ten successive, ordered stages, and each stage has two different possibilities (heads or tails). For both statements, the product that yields the total number of ways for "all the things to happen together" (Roberts and Tesman) is the same as the number of the total "different composite outcomes of the procedure" (Tucker). This "Coin Flips" problem is one in which both types of statements can be applied and yield the correct answer to the counting problem. The number of ways to complete the procedure is in a one-to-one correspondence with the number of desirable outcomes.

However, not all counting problems can be solved in such a straightforward manner. To detail our discussion of this issue, we turn to a "Words" problem presented by Tucker (2002, p. 172): *How many ways are there to form a 3-letter word using the letters a, b, c, d, e, and f, if the word must contain e and repetition of letters is allowed?* In applying an operational statement, we note that the first "thing" that will happen is to decide where to put the *e* that must be in the password, and there are 3 choices for this (the first, second, or third position of our word). The second "thing" is to choose a letter for the leftmost available position, and there are 6 choices, because repetition of letters is allowed. Then, the third thing to do is to choose a letter for the last remaining position, and again there are 6 choices. By an operational statement of the MP, then there are 3*6*6 = 108 ways of completing the process. This is, in fact, true, and there is no claim being made about what this means in terms of distinguishable desirable outcomes.

However, a key aspect of counting is that there is a relationship between a counting process and outcomes associated with that process (Lockwood, 2013). In this "Words" problem, it is true that there are 108 possible ways to complete the three-stage counting process. However, the outcomes of that process are not all distinct: notice that many of the outcomes – those involving 2 or 3 *e*s – would appear more than once in the list of 108 ways to complete the process. Our counting process generated some of the same outcomes more than once. If we simply wanted to count possible ways to complete a procedure, this would not be an issue. However, counting involves specifying the cardinality of the set of outcomes – determining exactly how many of something satisfies certain constraints. Therefore, the fact that in using an operational statement, we are counting ways to complete the procedure, and not actually determining the number of distinct outcomes, is problematic. By only counting ways of completing a counting process, without tying that to outcomes, there is a danger of overcounting when the ways of completing that process are not in one-to-one correspondence with the desirable set of outcomes.

Discussion and Conclusion

By drawing on Sfard's (1991) work in identifying and describing structural, operational, and bridge statements of the MP, we have demonstrated the different ways that the MP may be presented. Operational statements frame the MP as counting the *number of ways* to complete some process, procedure, or sequence of tasks. To us, this reflects viewing the act of counting as involving completing counting processes, but not necessarily about determining the total size of a set of outcomes. Lockwood (2014) has previously demonstrated the value of what is called a *set-oriented perspective*, which frames counting as being about determining the cardinality of a set of outcomes. This stands in contrast to how many of the operational statements of the MP situate the activity of counting. We feel that our findings suggest that, in fact, counting is not always framed as inherently involving counting sets of things, and structural and bridge statements might more naturally align with a set-oriented perspective.

Finally, a major pedagogical implication of our study is that the MP is much more nuanced than instructors and students give it credit for. Given its foundational place in counting, we need to help students focus more on understanding the details of the MP. Because there are clearly a variety of ways to present and talk about the MP, we feel that teachers of counting need to be very explicit with students about what exactly the MP is saying. Instructors could offer multiple statements of the principle and have a clear discussion of what a given statement in terms of ways of completing a counting process versus determining number of distinct outcomes of that process, as we discussed. In addition, instructors should very clearly explain how overcounting can occur in counting situations that involve multiplication. Regardless of which type of statement a student (or an instructor) prefers or which statement their particular book uses, students must be faced with the potential to overcount, and it may be up to the instructor to share this, especially if the book does not address it explicitly.

Endnotes

[1]We follow a number of authors by referring to the principle as the "multiplication principle" throughout the paper, even though the textbooks we surveyed had many different names for it.

[2]There are two ways in which we limited our search. We did not include universities outside of the United States to limit the scope and because we did not feel equipped to linguistically analyze textbooks in other languages. We also did not examine probability textbooks, again to limit the scope of the study, primarily because we suspect that reasoning about multiplication in probability contexts may fundamentally differ from strictly combinatorial contexts.

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