A MATHEMATICAL MODELING LENS ON A CONVENTIONAL WORD PROBLEM

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Given the Common Core's dual emphases on mathematical modeling, there is a need to understand modeling as a practice and content standard to develop students' mathematical modeling skills. This study of 12 students from differing levels of mathematics instruction and English Language proficiency includes analysis of their modeling with mathematics and a focus on their transitions through a mathematical modeling cycle. Findings suggest that students were engaging in critical processes that support mathematical modeling. We posit that conventional word problems can augment the benefits of using mathematical modeling tasks and can help educators explore a process-oriented approach to mathematical modeling.

Keywords: Modeling; Standards; Cognition

Globally, research in the teaching and learning of mathematical modeling spans the past 40 years. In the US, the models and modeling perspective (Lesh & Doerr, 2003) has given rise to model-eliciting activities (MEAs) (Lesh, Hoover, Hole, Kelly, & Post, 2000), model-development sequences (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003), and a design approach to developing these sequences. These research programs have focused on developing novel classroom instructional tools in order to teach significant mathematical concepts and on providing transformative teacher professional development.

Elsewhere, other mathematical modeling research theorizes the modeling *process* carried out by the modeler. The modeling process transforms a real world problem into a well-posed mathematical problem that can be analyzed mathematically. The results are then interpreted in terms of real world constraints and the model is validated. The mathematical model, or its representation in conventional mathematical terms (e.g., equations, graphs, etc.) is then iteratively refined. One such widely adopted mathematical modeling cycle (MMC) was posed by Blum & Leiß (2007) and has been used as a framework for examining and developing students' modeling skills It also serves as the basis of the CCSSM's description of mathematical modeling (CCSSM, 2010). Two complementary perspectives on mathematical models in the secondary curriculum are offered in the Common Core: one with the Standard of Mathematical Practice #4, Modeling with Mathematics and one with mathematical modeling as a Standard for Mathematical Content. Missing in the current literature is research on how to link the research findings (models of students' mathematical modeling) to the daily practice of solving conventional word problems in the secondary classroom.

The goal of this paper is to offer insights on students' mathematical thinking generated from a *process*-oriented view of their work on mathematical modeling tasks. It is a timely contribution as the mathematics education research community embarks on emphasizing mathematical modeling in the K – 12 U. S. classrooms. We choose to begin from the stance that the status of the task and the modeler's work is determined by the research lens rather than intrinsic properties of the task and we pose the question: What does a mathematical modeling perspective on a conventional word problem afford us?

Mathematical Modeling Cycle

The MMC is a description of the modeling process in terms of stages of model construction and modeling activities that are transitions between the stages. (See Figure 1.) The MMC was adopted as a research framework and focus was on the observable mathematical activities underlying each of the transitions in order to understand what a process view reveals about students' mathematical modeling. The MMC was operationalized via an observational rubric which conceptualizes each of the six transitions as a suite of mathematical activities (Czocher, 2013). The rubric was developed, refined, and validated via the method of constant comparison. Table 1 summarizes the transitions, the process they capture, and a sample indicator from the rubric. We offer an analysis of student work in the results section below. It is important to note that students may not exhibit all stages and transitions in order, or at all. Focusing on the transitions is intended to draw attention to mathematical thinking and activities being carried out, not to serve as a checklist of requirements.

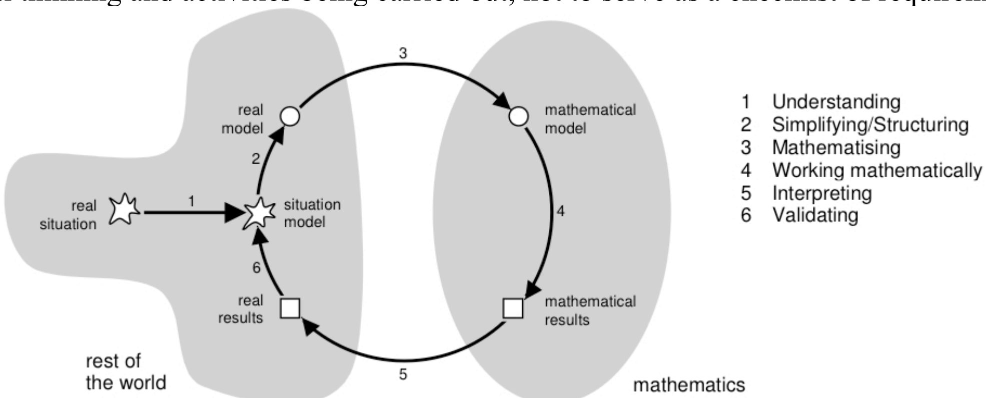

Figure 1 Schematic for a mathematical modeling cycle (Blum & Leiß, 2007)

Table 1 Modeling cycle transitions and sample indicators from the observational rubric

Activity	Trying to Capture	Sample Indicator
Understanding	Forming an initial idea about what the problem is asking for	Reading the task
Simplifying/structuring	Identify critical components of the mathematical model; i.e., create an idealized view of the problem	Listing assumptions or specifying conditions
Mathematizing	Represent the idealized model mathematically	Writing mathematical representations of ideas (e.g., symbols, equations, graphs, tables, etc.)
Working mathematically	Mathematical analysis	Explicit algebraic or arithmetic manipulations
Interpreting	Recontextualizing the mathematical result	Speaking about the result in context of the problem or referring to units
Validating	Verifying results against constraints	Implicit or explicit statements about the reasonableness of the answer/representation

Methodology

This study draws on a larger qualitative research design project aimed at understanding students' mathematical modeling processes. Participants were 12 middle and high school students who were selected so that four each were pre-algebra (6th grade), algebra (9th grade), and post-algebra (two in 10th grade; two in 12th). From each mathematical level, two had at some point in the U.S. schooling been identified as an English Learner (EL). All performed at the satisfactory or advanced level on the Texas mathematics standardized exam (STAAR) in the most recent year taken. 81% of the students at the middle school and 27% of the students at the high school that participated in this study are eligible to participate in the free or reduced price lunch program. The sampling plan was developed to be inclusive of mathematical approaches and representative of the diverse student population in

Texas. Tasks were provided in Spanish for EL students if they preferred. All of the EL students elected to conduct their interviews in English and explained their thinking in English. Due to the research design, this study reports qualities of students' modeling processes rather than an exploration of student characteristics.

The students participated in a series of three semi-structured, task-based interviews. For this study we consider their work on a conventional word problem, attempted by all 12 students, and similar to problems that appear in the algebra curriculum and on standardized tests. The students were asked to solve the following Turkeys & Goats problem (T&G): *A nearby farm raises turkeys and goats. In the morning, the farmer counts 48 heads and 134 legs among the animals on the farm. How many goats and how many turkeys does he have?* The task has value as a cornerstone of the institution of classroom algebra, but we do not claim that T&G is itself a modeling task. It is best classified as a concept-then-word problem (English, 2010) because it is designed and assigned to encourage students to practice already-learned procedures. T&G has the potential to reveal student thinking about making and justifying assumptions (e.g., one-to-one correspondence between heads and animals), working with real-world-imposed constraints (e.g., only whole-number animals should be considered), creating representations (e.g., equations or algorithms), and validating the resulting mathematical model. The ubiquity of similar tasks in mathematics classrooms suggests that analysis of student work via a mathematical modeling lens may be valuable in helping educators identify mathematical modeling processes carried out by students and therefore using such tasks to help students develop modeling skills.

The objective of each interview was to elicit the students' mathematical thinking and reasoning about the task, not to teach mathematics nor to teach mathematical modeling. The students were reassured that we were interested in their responses and explanations and not in whether the obtained the correct answer. Interviewer interjections were kept to a minimum. The interview sessions used design research principles of *cross-fertilization* and *thought experiments* (Brown, 1992). Cross-fertilization is when information and experiences from one interview session inform interviewer sensitivity and follow-up questions in another session. The interviewer posed clarifying and follow up questions ("Can you help me understand what you did here?") or asking the student to think aloud ("What are you thinking about?") or to provide additional reasoning ("Can you say more?"). Another set of interventions could be classified as scaffolding questions. For example, the interviewer adjusted the numbers in the problem if they proved too large for the student to reason about or calculate with. Thought experiments pose *what-if* questions that tweak the task as follow-ups to student explanations. For example, the interviewer could change the kinds of animals present on the farm so that the number of legs could not be realistically distributed among the animals.

Interviews were audio and video recorded and transcribed. The transcripts were summarized holistically according to the following dimensions: what approach did the student use, how it was executed, and what result was obtained. We used the observational rubric calibrated to the transitions in the MMC to look for evidence that the students were engaging in mathematical thinking that supported mathematical modeling (which is described below). When a mathematical activity was observed in the transcript or in the student's writing, it was tagged with a descriptive indicator and then coded with the associated transition from the modeling process. For example, if the student was observed to be carrying out a mathematical activity that could be described as an *explicit algebraic or arithmetic manipulation* that segment of transcript was coded *working mathematically*. In this way, the transcripts and students' written work were microanalyzed according to the MMC. Coding was carried out individually and then in pairs. All discrepancies were resolved. We then conducted a cross-case analysis looking for patterns in the students' modeling and to generate a list of questions and insights that arose from the two-layer analysis. This list guides our discussion of the implications of using a mathematical modeling lens on tasks currently used in mathematics K-12 classrooms.

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Results

A solution to the Turkeys & Goats problem satisfies two conditions simultaneously (a fixed number of heads and a fixed number of legs). An algebraic solution strategy was defined as an attempt to write symbolic, algebraic expressions describing the relationships among the total number of heads and the total number of legs. A partitioning strategy was defined as some intention to separate the total number of heads into goat heads and turkey heads (or separate the total number of legs into those belonging to goats and those belonging to turkeys) and work out how many legs (heads) must belong to each group. The 6 students who attempted a partitioning strategy began with a halving strategy that there were 24 goat heads and 24 turkey heads. Based on typical student explanations, this is because there were two kinds of animal and $48/2 = 24$.

The partitioning strategy led to a guess-and-check approach where the number of heads (legs) was adjusted based on the outcome from the previous assumption. This approach handles the two conditions sequentially and iteratively. The strength of the algebraic approach is that it generalizes the guess-and-check strategy by simultaneously evaluating the cases that arise and so it is more mathematically efficient. However, using an analytic, algebraic approach requires that both conditions are made explicit via the implicit assumption that each animal has one head and the constraints that goats have one head and four legs and turkeys have one head and two legs. The modeling lens recognizes that this transformation is nontrivial and it allows us to decompose students' difficulties in formulating mathematical constraints. Regardless of student level, the task has the potential to afford insights into their mathematical thinking. Table 2 summarizes the students' work on the task according to mathematical level and EL status.

Table 2 Summary of student strategies and answers

Level	EL	Algebraic Strategy	Partitioning Strategy	Both Constraints	Any Answer	Correct Answer
Pre	EL	0	2	1	2	0
	Non EL	0	1	1	2	1
Algebra	EL	0	1	1	1	0
	Non EL	2	0	1	2	1
Post	EL	1	1	1	1	1
	Non EL	1*	1*	1	2	1

* The same student used a partitioning strategy and then transitioned to an algebraic strategy.

At each level there was at least one student who executed neither solution strategy. Of the 9 students who attempted either strategy, only 6 recognized the need to satisfy both constraints at once. This discrepancy may point to a reason why the other 3 students were unable to use a solution strategy and they may have been unable to see a way to handle both constraints simultaneously. Ten of the 12 students obtained an answer but only 4 obtained a correct answer. For the pre-algebraic students, this may not be surprising since solving the task without algebraic techniques is cognitively demanding and computationally inefficient. However, of the 8 students who were in algebra or post-algebra classes, this is quite surprising given that such systems-of-equations tasks are a focus of instruction and high stakes tests.

Due to space constraints, we present a synopsis of one student's work on the Turkeys & Goats problem. Bree was a pre-algebra, non-EL student. Her work was selected because she developed a non-algebraic of the situation and because she focuses autonomously on satisfying both constraints. Thus her work illustrates what the students, regardless of mathematics level, are capable of doing. Transitions from the MMC are marked in the first part of the synopsis to demonstrate how the MMC is present in the student's work. Though we highlight one student's work, all students exhibited

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transitions from the MMC, demonstrating that they were engaging in mathematical modeling when working on the task even if no answer or conclusion was ultimately reached.

Bree began by reading the problem aloud (*understanding*) and then questioned the number of legs on a turkey (*simplifying/structuring*). She initially tried 48×4 (*mathematizing, working mathematically*) to obtain the total number of legs (*interpreting*) because "it'd be easy." She abandoned this approach because "some of them have two legs" (*validating*). She continued, "since there's turkeys I have to figure that out. So it's either that most of them could be like, I'm going to try half" (*simplifying/structuring*) and decided to find the number of legs among 24 turkeys and 24 goats by combining 24×2 with 24×4 (*mathematizing*) and using the standard algorithm (*working mathematically*) to get 144 legs (*interpreting*). Bree noted that she needed 134 legs. She wondered what to do about the "10 extra" legs. The interviewer asked her what she was thinking. She responded "What I was thinking first is if I, maybe if I div…cut the 48 in half, 24 turkeys and 40, 24 goats, then I figure I would, so 48 legs for the turkeys and 96 legs for like goats…I got 144. But I think we needed 134 legs, so I'm trying to figure out right now how I could put that down." Bree switched her focus from the heads constraint to the legs constraint. As a follow up, the interviewer asked how she would reduce the number of legs. She concluded that there would have to be more goats. Next, Bree tried 22 turkeys and 26 goats and realized that she added legs. She reversed her guess and tried 22 goats and 26 turkeys to obtain 140 legs. She shook her head because she recalled that "We're trying to find 134."

Bree changed her approach yet again: "So I guess I could try to find how many times 4 goes into 134 or how many times 2 goes into 134." She proceeded to carry out long division via the standard algorithm for $134/4$ and $134/2$. She obtains 32 goats and 6 turkeys, but did not realize that she was missing 13 heads. She became confused on how to figure out the number of legs for a given number of heads and backpedals to her computation $134/2.$ She states that this would yield 67 turkeys and no goats which is impossible because there were only 48 animals. At this point, Bree announced "There has to be an answer" and began working quietly doing various multiplicative computations.

The interviewer asked, "Could you summarize for me what you know about what the possible combinations [of turkeys and goats] might be?" This prompts Bree to organize her work into a list of goat/turkey combinations she had tried and begin systematically adjusting those values. She stated her strategy, "I've tried 22 goats and 26 turkeys and I got 50, 52, and 88 goats, but that equaled 140. So that one didn't work. And 9, 8, I guess I could try other ones just like that. I just have to keep on going down 'til I got to 134." She used her algorithm developed from trying 24 goats with 24 turkeys and 22 goats with 26 turkeys to "keep on going down" until she found the combination 19 goats with 29 turkeys yielded 134 legs. When asked to explain how she knew to increase the number of turkeys and decrease the number of goats, she responded that "4 doesn't go into many numbers as much as 2 does 'cause it's bigger," indicating that she had some sense of needing the net number of legs to decrease.

Implications and Conclusions

The Turkeys & Goats problem is a conventional word problem similar to those found in algebra textbooks and on standardized tests. We presented two layers of analysis on students' work: one focused on solution obtained by the student and how the student carried it out and one that examined students' mathematical activity in support of mathematical modeling processes. The first is a product-oriented perspective that focuses on representation of the solution and its correctness. According to this perspective, many of our students were unable to solve a conventional word problem. However, this kind of surface level analysis is limited in that it does not reveal the deeper complexities of student's mathematical thinking and understanding which are useful for informing classroom instruction. A closer examination of students' mathematical thinking showed that students

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took an algebraic approach or used a partitioning strategy. Analysis of student strategies reveals how they were thinking about the task.

The process-oriented perspective focused on mathematical modeling as a process. Operationalizing the MMC in terms of mathematical activity and then applying the rubric to students' work showed that all students were engaging in mathematical modeling processes. Analysis revealed how students progressed through the task even if they did not ultimately obtain a right (or any) answer. Bree's work demonstrated that students are capable of modeling with mathematics, a Standard of Mathematical Practice (CCSSM, 2010) without necessarily using an explicitly algebraic approach. She carried out a guess-and-check strategy, but her guesses were not haphazard. She developed an algorithm based on constraints implied by the problem statement in order to relate the number of heads of each kind of animal to the total number of legs among the animals. To do so, she introduced two parameters: the number of turkey heads and the number of goat heads. After each trial, she adjusted her estimates in ways that anticipated their effects on the outcome (number of legs). That she was capable of organizing her work into a systematic algorithm after some prompting and encouragement suggests that she was at a participatory stage of forming a concept about satisfying multiple constraints simultaneously but that she had developed an activity-effect relationship between adjusting the head parameters and verifying the number of legs (Tzur & Simon, 2004). In essence, she was able to apply her mathematical model as an algorithm dependent on input parameters.

Thus a product-oriented perspective on a conventional word problem can impede educators from fully understanding students' mathematical activity because it emphasizes what students could accomplish. Therefore, it reveals student struggles in achieving objectives in mathematics instruction. In contrast, a process-oriented perspective on mathematical modeling offers more information that educators can base decisions on because it shifts attention to the students' mathematical activity and allows for articulation of what students are capable of doing. The shift has implications for helping classroom mathematics teachers identify and harness student success in mathematical modeling in a way that is grounded in students' current mathematical activity rather than solely on obtaining a correct answer.

Our analysis suggests that independent of where the student is in their formal mathematical instruction, they are capable of and spontaneously do exhibit the kinds of thinking that support mathematical modeling. These results parallel the findings of Cognitively Guided Instruction (CGI) research: children do not need explicit direct instruction in modeling (Carpenter, Fennema, Franke, Levi & Empson, 2015). Children come to school with ways of thinking about mathematics that do not need to be formally taught. Instead, the goal is to understand their thinking and original strategies and guide them toward more efficient and proficient ways of solving problems. Likewise, the goal in teaching modeling should be providing opportunities for the students to develop their judgment to become comfortable making assumptions that will satisfy a complicated situation. As demonstrated by Bree, the students may already have a sense of needing to validate their models and revise their assumptions. Reinforcement of the idea that assumptions and representations may require revision could be provided by working on techniques for validation.

Thus, classroom mathematics teachers may not need to consider the modeling standard and practice articulated by the CCSSM as something "new" to add to the curriculum. Our analysis shows that even on conventional word problems, students are already thinking in ways that support mathematical modeling. The challenge becomes helping teachers identify what the modeling process looks like so that they can recognize when a student is developing a model (not just that the strategy is inefficient) and ask specific questions of students at specific times. A comparison between process- and product-oriented views of students' mathematical activity can help educators focus on mathematical structure instead of just correcting mistakes in carrying out procedures. Such emphasis may also push students toward process-oriented views of their own mathematical activity instead of focusing on obtaining the correct answer. .

Using the MMC to see beyond solution strategies on a seemingly straightforward task led to conversations about the role of the interviewer (and by extension, the role of the teacher in a classroom setting). Though all students were capable of making progress in the modeling process, the realization that the models needed revision and some encouragement as to how to do so was influenced by interviewer questions. In Bree's case, to fully develop her algorithm she was prompted to summarize what she had done so far. Perhaps realizing that their models may need revision later contributed to some students' unwillingness to commit to a solution strategy. Future research must critically examine the role of interviewer prompts in scaffolding mathematical modeling. Such analysis would have implications for when and how teachers may most productively intervene in students' modeling processes while respecting the students' ideas and model development. In addition, due to the ELs opting to conduct their interviews in English, we can continue to question and think about the implications of past schooling on their mathematical modeling. One of the challenges in implementing the CCSSM has been that teachers may not share the same vision of how to operationalize the standards in the classroom as the standards writers intended. There is a need for research-based understandings of what teachers may be currently doing in classrooms which may need to be reconsidered or adjusted in order to fully realize a mathematical modeling perspective. We suggest that mathematical modeling need not be a wholly new undertaking. The participants in this study demonstrate that K-12 students are engaging in mathematical modeling processes whether it is taught explicitly or not. The challenge becomes helping teachers identify it – it's not foreign -- and how their students are doing it.

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