"NATURAL RESOURCES": TWO CASE STUDIES IN EARLY EXPRESSIONS OF GENERALITY

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In this paper, we analyze individual semi-clinical interviews conducted with one kindergarten and one first-grade student. We build on prior research to offer evidence, via excerpts from these interviews, that children as young as kindergarten have a powerful, intuitive sense of generality and indeed naturally draw upon it to reason through mathematical scenarios. We identify within these children's utterances four features of generalizing for which educators might attend.

Keywords: Classroom Discourse; Algebra and Algebraic Thinking; Elementary School Education

Introduction, Issues, Theoretical Framework

Hyman Bass credits generality as "one of the most important and powerful characteristics of mathematics" (2003, p. 326); John Mason dubs it mathematics' "heartbeat" (1996, p. 65). Concordant with these voices, both the *Common Core State Standards for Mathematics Initiative* (CCSSI) and the National Council of Teachers of Mathematics' (NCTM) *Principles and Standards for School Mathematics* emphasize generalizing as a key mathematical practice throughout the grades, one that can and should be encouraged early and honed over time (CCSSI, 2010; NCTM, 2000). To truly foster this practice beginning in the early grades, however, educators must first recognize young students' generalizations, a task complicated by the extent to which the language children employ differs from that of adults. The risk of underestimating the robustness of children's understandings or overlooking their insights altogether may be especially pronounced in the domain of mathematics, whose exactness can lend itself to particularly formalized conventions as to how generality should be expressed. For instance, universal qualifiers, be they explicit or implicit — "for all $x \geq 0$, $|x| = x$," "multiples of four are even," or "every integer greater than 1 has a unique prime factorization" — are the standard fare of generalizations in mature mathematical discourse. As students are inducted over time into communities of mathematical discourse, they gradually assimilate these conventions of communication. But there is no reason to believe that children in the earliest grades would have learned these communicative conventions, and there is no simple litmus test for whether children are thinking in generalized terms. These challenges call for further research into the range of ways in which very young children may express generality. By honing our own ability to recognize early inclinations to generalize, we can, in turn, better nurture their development.

Identifying young children's generalizations can be challenging, requiring "the skilled and attentive ear of a teacher who knows how to listen carefully to children" (Kaput, 2000, p. 6). Our goal in this paper is thus twofold: (1) to offer evidence, via excerpts from mathematical interviews with two case students, that children as young as kindergarten have a powerful, intuitive sense of generality and indeed naturally draw upon it to reason through mathematical scenarios, and (2) to identify within these children's utterances several characteristics indicative of generalizing for which educators might, in Kaput's phrase, "listen carefully."

Prior research has yielded numerous examples, largely from teacher documentation or classroom discussions, of young children's generalizing about comparison of quantities (Schifter, Bastable, Russell, Riddle, & Seyferth, 2008a) including non-specified quantities (Dougherty, 2008), commutativity (Schifter, Monk, Russell, & Bastable, 2008b), classes of numbers (Bastable & Schifter, 2008), additive and multiplicative identity properties (Carpenter & Levi, 2000), and use of the equal sign (Carpenter, Franke, & Levi, 2003). As a body, this research demonstrates that it is within the reach of young children to observe mathematical regularities and talk about their discoveries in a variety of ways.

What does it mean, really, to generalize? Rowland (2000), Carraher, Martinez, and Schliemann (2008), and Kaput (2000) each propose perspectives. According to Rowland (2000), generalizations are statements of beliefs about properties of an entire class that have not and indeed cannot be inspected and tested. For Carraher et al. (2008), generalizations involve claims for an infinite number of cases, where "the *scope* of the claim is always larger than the set of individually verified cases" (p. 3, italics in original). Finally, Kaput (2000) highlights that generalizations "deliberately extend … the range of communication beyond the case or cases considered, [to the] patterns, procedures, structures, and the relations across and among them" (p. 6). These three proposals share an emphasis on the generalizer's conviction with respect to an inference that includes many cases simultaneously, a conviction that obtains in the absence of envisioning each of those individual cases.

Mode of Inquiry, Data Sources

In this paper, we analyze individual semi-clinical interviews conducted with one kindergarten and one first-grade student as part of a research project focused on exploring kindergarten through second-grade (K-2) children's understandings of functions. The data are drawn from an eight-week classroom teaching experiment (CTE). Individual interviews with a subset of students in each of the three grades were carried out immediately prior to, halfway through, and at the end of the CTE. The students in the school in which the CTE was conducted are 98.6% minority (non-white), with 89.5% categorized as low SES and 33.9% as ESL.

To facilitate the process of data reduction, a series of steps was taken. First, because the goals of the study were to explore young children's intuitive sense of generality and to provide evidence for what generalizations look like among young students, we focused on the initial interviews (pre-interviews), which took place before any lessons were implemented. By selecting for analysis data collected before the teaching experiment began, we ensured that each child's productions, both verbal and written, would best approximate what might be described as "naturally" present. Second, we focused our selection on kindergarten and first-grade students to explore expressions of generalization among the youngest children in our study.

In our pre-interview protocol, students were first asked how many noses a dog has. They were then asked how many noses there would be altogether among two dogs, three dogs, and so on. We did not doubt that students would answer correctly for all specific cases proposed. Rather, we were interested in students' explanations of how they knew — that is, in how they spoke about their reasoning. Each student was asked at some point in the interview to organize the information in a function table, reason about far values (e.g., the case of one hundred dogs), and reverse the relationship (supply a number of dogs given a number of noses), as well as to respond to a proposed "mismatched" case (e.g., the suggestion of five dogs and six noses).

Students were also asked how they might tell a friend how to know the number of noses for any number of dogs and whether there was a rule for making this determination. They were also invited to generate their own examples rather than simply responding to interviewer-generated values. These questions were intended to create an open-ended space for students to verbalize their understanding of the problem and the functional relationship that governed it.

All interviews were transcribed verbatim from video, and all video and transcripts were reviewed. This review facilitated a progressive selection of the dataset for this paper. Our criterion

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for selecting the students for this paper's analysis was that they exhibit diverse verbalizations of mathematical ideas. Our aim was not to showcase the most sophisticated thinking in young students, but rather to highlight a range of ways young students' mathematical thinking might find expression. In this way, we selected interviews with Kinetta and Ferdinand, a kindergartener and first-grade student respectively.

In analyzing these interviews for generalizing, we appealed to the common thread among the perspectives on generalizing noted above: conviction in an inference about many simultaneous cases that is independent of envisioning of those individual cases. We reviewed the transcripts line-by-line using the constant comparative method (Glaser & Strauss, 1967). Our research goal was to identify features of this thread within these students' interviews.

Results

Bills (2001) theorizes that qualitative differences in students' language correspond to qualitative differences in their conceptual constructions and that these shifts in language may be markers of progress towards recognizing the generality of procedures. Adopting this premise, we highlight four prevalent features of the two case students' verbal productions that emerged as a result of our analysis. Examples of each of these features will be provided below in Table 1.

- *Definite Articles, Indefinite Quantities*: We observed that when students were asked questions that might easily have been construed as centering on a particular case, they often nonetheless replied with a generalized answer that could accommodate any case. In such instances, the student also indicated that a strict rule would uniquely determine the relevant numerical "output" based on the input, whatever the input might happen to be. We took these instances, referred to here as "definite articles, indefinite quantities," to indicate that the student was spontaneously generalizing, displaying conviction about many simultaneous cases not individually envisioned.
- *Certain Denial*: If a student was prepared to cry foul without hesitation in response to a mismatched (e.g., five dogs and six noses) scenario proposed by the interviewer and to justify and defend his or her position, we took this as an indication of the student's conviction. (These impossible scenarios were akin to Carpenter and Levi's [2000] and Davis' [1964] false number sentences, leveraged as windows into young children's ability to justify generalized properties of whole numbers.) Moreover, if the student justified his or her conviction by giving reasons that appealed to the general logical structure of the problem, as opposed to simply the particular case, we took this as evidence of generalized thinking about cases not individually envisioned. We refer to this feature as "certain denial."
- *Generic Examples*: We adopt this terminology from Balacheff (1988), who lists generic example as the third of four main forms in the cognitive development of proof. Balacheff makes much of the transition from the second form — termed "crucial experiment" — to its successor. While a crucial experiment offers only the outcome of a particular case to support a general conclusion, the case has been chosen deliberately for its perceived particular ability to carry that import.[1] Balacheff maintains that one crosses a "fundamental divide," or undergoes a "radical shift in ... reasoning" in stepping from crucial experiment to generic example, and that in the latter territory, one "establish[es] the *necessary nature of [a] truth* by giving reasons ... by means of operations or transformations on *an object that is not there in its own right, but as a characteristic representative of its class*" (pp. 218-19, emphasis added to correspond to conviction with respect to many simultaneous cases not individually envisioned). Accordingly, in our study, we took students' generic examples to be evidence of

generalizing. Bills (2001) also regards generic examples as steps toward more formal generalizations.

- *Authoritative "You"*: Rowland (1992, 1999, 2000) has paid considerable attention to children's use of the pronoun "you." One use is as a substitute for the more formal "one" and thus as an indicator that a generalized procedure is being described. Rowland notes that this procedural "you" is common even in non-mathematical situations, especially but not only among children — for instance, in explaining how to play a game. He has observed that a shift from "I" to "you" in children's discourse seems to parallel a shift from explaining work done with specific cases towards describing a more generalized procedure. This procedural "you" also appears in excerpts from Schifter et al. (2008a), Bastable and Schifter (2008), and Carpenter and Levi's (2000) research with young children (though these researchers do not highlight it as such in these studies): for instance, "you don't have to pay attention to the 6s," "each time you add a number to a group that can go, you get a group that can't," and "when you put zero with one other number, just one zero with the other number, it equals the other number," respectively. A second use of "you," distinct from the first, is as a pronoun of direct address. Rowland notes that it is considerably less common for students to use "you" in this fashion when speaking to teachers than vice versa, an imbalance he attributes to power relations. Thus we take students' usages of direct-address "you"s to be indicative of their willingness to assume an authoritative position and their usages of procedural "you"s to be indicative of explaining generalized rules or procedures — which in turn we take as indicative of conviction with respect to many simultaneous cases not individually envisioned. We refer to this feature as "authoritative 'you.'"

Owing to space constraints, we give limited excerpts from each case student's interview and highlight these four features within the transcript. We use **boldface** text to foreground particular phrases that, situated in context, exemplify these features. Following the transcripts, we organize these examples in a table (see Table 1) with some discussion.

Excerpt from Initial Interview with Kinetta (Kindergarten)

Interviewer: If there are three dogs?
Kinetta: Three noses.
Interviewer: How do you figure out? How do you know how many noses there are?
Kinetta: You count.
Interviewer: How do you count?
Kinetta: One, two, three.
Interviewer: Mm-hm. And how do you know how to stop — when to stop counting?
Kinetta: When **you** get to **the** number.
Interviewer: So [...] for instance, what if there were ten dogs? How many noses [...]?
Kinetta: Ten.
Interviewer: How do you know that?! I didn't see you count! I didn't see you do any counting.
Kinetta: That's because I counted in my head.
Interviewer: Oh! You went all the way to ten that quickly?
Kinetta: [Nods.]
Interviewer: [...][2] What if there are one million dogs? How many noses are there?
Kinetta: One million.
Interviewer: You did not count that fast. How do you know what number to say?
Kinetta: Because **you** just said **it**.
Interviewer: Oh! What do you mean?

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Kinetta: **You** said "one million."
[...]
Interviewer: What if there were twenty-four noses?
Kinetta: Twenty-four dogs.
Interviewer: Was that easy? Yeah? How did you figure that out?
Kinetta: Because **you** just said **it.**
[...]
Interviewer: Can I show you something that some people use? It's called a table. [Draws function table setup.] They do this, and here they put th— how many dogs, and here they'll put how many noses. So for instance, if it's one dog [writes the numeral 1 in the left-hand column of the function table], how many noses?
Kinetta: One.
Interviewer: [Writes the numeral 1 in the right-hand column.] If it's two dogs [writes 2 in the left-hand column]...
Kinetta: Two noses.
Interviewer: And can you put it? Can you show it right there [in the right-hand column]?
Kinetta: [Writes 2 in the right-hand column of the table.] [...]
Interviewer: What if there were three here [in the left-hand column]? How many noses?
Kinetta: [Writes 3s in both columns.]
Interviewer: Oh, great, and what if there were— let's put another number here [in the left-hand column]. Whatever number you want.
Kinetta: **[Writes 100, 100.]**
Interviewer: Oh my goodness! What number is that?
Kinetta: **One hundred.**
[...]
Interviewer: If we put a number here, whatever number here [in the left-hand column], what number do we have to put here [in the right-hand column]?
Kinetta: **The same one.**
[...]
Interviewer: So, you know what someone told me? Someone told me that there were five dogs and there were six noses. What do you think of that?
Kinetta: **[Shakes head no.]** [...]
Interviewer: No? Why not?
Kinetta: **Because it needs to be the same.** [...] Six noses and six dogs. Five noses and five dogs.

Excerpt from Initial Interview with Ferdinand (First Grade)

Interviewer: So, if — what if we put here that there's one dog, okay [writes the numeral 1 in the left-hand column]? How many noses are there gonna be?
Ferdinand: One.
Interviewer: One [writes the numeral 1 in the right-hand column]. What if there are two dogs [writes 2 in the left-hand column]?
Ferdinand: It's two.
Interviewer: Okay, can you show me that?
Ferdinand:[Writes 2 in the right-hand column.]
Interviewer: What if there are three dogs?
Ferdinand: Then three [writes a 3 in each of the two columns].
Interviewer: Do you want to show me some other numbers of dogs?

Ferdinand: We're going like number?[3]
Interviewer: Whatever you want. If you want to do it that way, we can do it that way.
Ferdinand: **I'm gonna do a five [writes 5 in the left-hand column of the table].** [...]
Interviewer: What would you put on the other side? [...]
Ferdinand: Oh, five [writes 5 in the right-hand column]. [...]
Interviewer: How do you know it's five noses?
Ferdinand: 'Cause fiv— it's five 'cause five noses **has to** be the same, **like** — they could play, **if** they didn't have a lot of— **like** they're playing hide and seek **if** they didn't have a lot of people to play—
Interviewer: Mm-hm.
Ferdinand: **So they have to** have five and five.
Interviewer: Mm, okay.
Ferdinand: **Like** teams.[4]
Interviewer: So, so if a friend asked you what number — how you know what number to put here, what would you tell them?

Table 1: Examples of the Four Features

Feature	Key phrases (regarded in context)	Discussion
Definite Articles, Indefinite Pronouns	"When you get to **the** number" (line 8); "Because you just said **it**" (line 19, line 26); "**The same one**" (line 48). "**It** was **the** number that — that you said" (line 82).	Lines 8, 19, 26, and 82: Rather than give answers specific to the particular cases the interviewer has just referenced — of three and five dogs, respectively — as they might reasonably do, both students instead respond in generalized terms even though they have not been "asked" to generalize. Line 48: Kinetta appears to have no trouble responding when posed a question in general terms.
Certain Denial	"**[Shakes head no]** [...] **Because it needs to be the same.** [...] Six noses and six dogs. Five noses and five dogs" (lines 52, 54-55). "You say, 'You **have to** take one more out, 'cause we **have to** have five and five, **'cause** [...] we **have to** play five-five, **'cause**[...]'" (lines 86-88).	Both children appeal to the logical necessity of sameness: it "needs to" or "has to" happen for a reason. Additionally, that Kinetta allows for both possible corrections of the mismatch (five-five and six-six) suggests an understanding that this sameness is not only necessary but also sufficient. She's willing to vary the number of dogs *or* the number of noses, but she insists that those two counts have to match.
Generic Example	"**[Writes 100, 100]** [...] **One hundred**" (lines 42, 44). "**I'm gonna do a five [writes 5 in the left-hand column of the table]**" (line 69); "It's five 'cause five noses **has to** be same, **like** — they could play, **if** they didn't have a lot of— **like** they're playing hide and seek **if** they didn't have a lot of people to play [...] **So they have to** have five and five [...] **Like** teams" (lines 73-75, 77, 79).	While the dogs-and-noses problem centers on an identity function, and thus in some sense masks the "transformations" enacted on objects, we maintain that these examples proposed by Kinetta and Ferdinand are nonetheless "characteristic representative[s] of [their] class," "objects ... not there in [their] own right" (Balacheff, 1988, pp. 218-219). In Kinetta's case, one hundred dogs is fundamentally a case not individually envisioned. While one might precisely visualize one, two, or three dogs, it's reasonable to surmise that it's fairly impossible to hold a quantitatively precise mental image of one hundred dogs. One hundred serves as a representative of the class of theoretically possible, potentially arbitrarily large quantities of dogs. Ferdinand's usage of "if" and "like" (lines 73-75, 79) resonates with Bills' (2001) findings that children often use "if it's like" to introduce generic examples (where adults, Bills posits, might use "for example," "consider for instance," or "suppose"), as well as with the diction that accompanied the use of generic examples in Carpenter and Levi's (2000) study. Furthermore, that Ferdinand says that "five noses **has to** be the same" (line 73) and concludes, "**So they have to** have five and

		five" (line 77) suggests that, as Balacheff describes, what he's arguing about is the "necessary nature of [the] truth."[5]
Authoritative "You"	"When **you** get to the number" (line 8); "Because **you** just said it" (line 19, line 26); "**You** said one million" (line 21). "It was the number that — that **you** said" (line 82).	Line 8: "You" is used impersonally to convey a generalized procedure. Lines 19, 21, 26, 82: "You" is used as a pronoun of direct address.

Ferdinand: "**It** was **the** number that—that **you** said."
Interviewer: Okay. [...]What if you had a friend who said that they counted that [...] with five dogs, there are six noses? [...] Five dogs and six noses [points to these places on function table]. What would you tell your friend?
Ferdinand: You say, "You **have to** take one more out, 'cause we **have to** have five and five, **'cause** [...] we **have to** play five-five, **'cause** if we don't have five-teams, we ha— we have more than five."
Interviewer: Okay.[...] If your friend says that there are ten dogs and he counted twelve noses?
Ferdinand: It'll be — twelve.
Interviewer: Twelve what?
Ferdinand: Dogs.

Significance of the Research

The examples we provide from interviews are far from exhaustive; nonetheless, they build a portion of a catalogue to which educators might look to identify instances in which students as young as kindergarteners are spontaneously generalizing, so that we might better capitalize on opportunities to foster this type of reasoning. Importantly, for our purposes, they contribute to an existence proof that young children do indeed bring natural intuitive powers of generalizing to formal schooling and that, as such, the power of generalizing is "natural, endemic, and ubiquitous" (Mason, 1996, p. 66).

Endnotes

[1]This contrasts with the first of the forms, naïve empiricism, which lacks the deliberate "*this* is my *test* case" element, in that it entails believing a proposition simply because one conceives of *a* case, or a handful of cases, that "works."

[2]Throughout the transcripts, the symbol [...] indicates dialogue removed for irrelevance.

[3]We take Ferdinand to be asking here whether he was expected to increment the number of dogs consecutively.

[4]We take the analogy to be to the necessity that two teams have equal numbers of players.

[5]Kinetta also speaks to the logical necessity of sameness when she says (line 54) "it needs to be the same." However, since this was an instance of her rejecting a proposed "impossible case" as opposed to proposing an example of her own, we have not considered it a generic example.

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