DUAL ANALYSES EXAMINING PROVING PROCESS: GROUNDED THEORY AND KNOWLEDGE ANALYSIS

Aditya P. Adiredja
University of Arizona
adiredja@math.arizona.edu

Shiv Smith Karunakaran
Washington State University
shiv.karunakaran@wsu.edu

This report presents dual analyses of an undergraduate student, Cassie, whose work provides nice contrasts between Grounded Theory (GT) analysis and Knowledge Analysis (KA). The analyses highlight particular methodological differences, such as grain size of findings, positioning of novices and more general implications about expert–novice studies. The combination of the two methods results in a more complete and nuanced description of Cassie as a prover, while mediating many of the methodological concerns from the individual analysis.

Keywords: Cognition, Equity and Diversity, Reasoning and Proof, Research Methods

At the higher academic levels (graduate and professional mathematics), proving can be considered to be a way in which the truth of a claim is established or realized (Hanna, 2000; Weber, Inglis, & Mejia–Ramos, 2014). Proving, and more generally, justifying is a process by which mathematical knowledge can be furthered throughout the K–12 grades and in higher mathematics education. Selden, McKee, and Selden (2010) stated that the proving *process* "play[s] a significant role in both learning and teaching many tertiary mathematical topics, such as abstract algebra or real analysis" (p. 128). Various empirical studies have ventured to provide insight into the process of proving separate from the product of proof. For instance, previous work involving the comparing of expert and novice productions of proof have examined issues such as differences in strategic knowledge in the construction of proofs in abstract algebra (Weber, 2001), private and public aspects of proof (Mejia–Ramos & Tall, 2005; Raman, 2002), and use of examples by doctoral students in evaluating mathematical statements (Alcock & Inglis, 2008). This current work adds to this corpus of literature in order to better understand the process of proving, without necessarily associating the process to the finished product of proof.

The study of the process of proving as an unfamiliar phenomenon benefits from the use of qualitative methods that focus on explaining the phenomenon being studied instead of validating existing theory about the phenomenon. Grounded Theory (GT) research is a qualitative research method that is defined by the generation of theory that is inductively derived from the study of data (Glaser & Strauss, 1967). GT research can be used to investigate how individuals go through a process and to identify the different steps in that process (Charmaz, 2000, 2006; Creswell, 2007). More specifically the types of questions routinely investigated using GT methods are of the type: "What was the process?; …What was central to the process? (core phenomenon); … What strategies were employed during the process? (strategies); [and] What effect occurred? (consequences)" (Creswell, 2007, p. 66).

The goal of a GT analysis is to develop a substantive theory based on categories that are generated from iteratively gathering and examining data until the categories generated have been sufficiently saturated. This often implies that any theory developed through GT analysis is expected to be coarser and may necessarily sacrifice details and nuances about the phenomenon. Parnafes and diSessa (2013) asserted that claims about cognitive processes using GT typically use "time–scale and meaning–resolution [that] are rather large and indefinite" compared to other types of analyses of learning that are more microgenetic in nature (p. 15).

One such method is Knowledge Analysis (KA) (diSessa, Sherin, & Levin, 2016; Parnafes and diSessa 2013). KA is a methodological approach associated with the Knowledge in Pieces (KiP)

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

theoretical framework (diSessa, 1993) to "study the content and form of knowledge for the purpose of understanding learning" (diSessa et al., 2016). KA aims to describe details of models for mental representations of students' knowledge through micro–assessments and tracking of an individual's learning in real time. In principle, KA explicitly rejects the notion that novices' knowledge is a subset of experts' knowledge, and thus prioritizes the investigation of how mathematical knowledge emerges out of naïve thoughts.

In this paper, we employ both GT analysis and KA to explore the proving process of Cassie (a pseudonym), a "novice" prover. We anticipate that the two methods to complement each other in some aspects but to diverge in others. The goal is to provide a more complete and nuanced account of Cassie's proving process with the two methods, and to simultaneously contribute to the more general discussion about combined methods in qualitative research.

Methods

Data Source

Data was drawn from a larger study (Karunakaran, 2014), which examined the similarities and differences in the usage of knowledge by expert and novice provers of mathematics. The larger study involved individual semi–structured interviews where 5 undergraduate mathematics students and 5 mathematics Ph.D. students validated or refuted the truth of five mathematical statements. At the time of data collection for the larger study, Cassie was a female undergraduate student majoring in mathematics. Cassie was selected because her work offers a palpable illustration of the affordances of using the individual methods of GT and KA. Cassie's proving work was not unique compared to that of others within the novice prover group. In fact, the use of the GT analysis necessitated that the resultant claims about Cassie were consistent with her peers. Five mathematical statements were presented to Cassie over the course of two 90-minute interviews, which were audio and video recorded for subsequent transcription and analysis. The analysis in this paper focuses on Cassie's engagement with the first task.

Grounded Theory Analysis

GT analysis uses methods such as *constant comparison*, and coding strategies such as *open coding*, *axial coding* and *selective coding* (Charmaz, 2000, 2001, 2006; Straus & Corbin, 2008). These phases of coding can be described as examining the data in order to develop categories of information (open coding), examining these categories to develop them further and to interconnect them (axial coding), and using these developed categories and their interconnections to build a theory that explains the existence of the categories (selective coding).

To instantiate this coding and analysis process, consider the following excerpt from Cassie's first interview, where she was addressing the task shown in Figure 1. For space considerations, the transcript has been abridged to exclude non–mathematical language and probing questions from the interviewer.

> *Cassie:* Let's see. So if a_n were 1, uh a_1 were 1 then that would have to be less than or equal to a_2, a_3 um so and then that would have to be um that would have to be less than or equal to a_4 plus a_5 plus a_6 plus a_7 /…/ Ok. (Pause) um I'm inclined to say that it's false. /…/ Because the, I mean the only constraint is like in the future so the a_{2n}. a_{2n+1} can grow arbitrarily large and like all that matters is that the next two, like the next a_{2n} and a_{2n+1} are bigger than that. … So like a_n is only constrained by a_{2n} and a_{2n+1}. But uh nothing before it so my initial inclination is to say that it's false.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

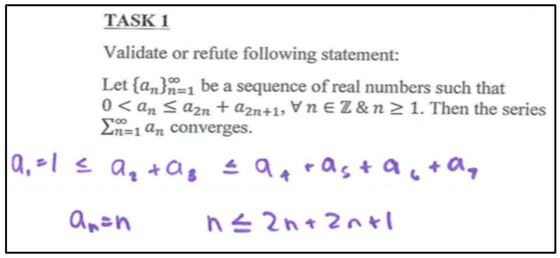

Figure 1: The task statement for Task 1 and Cassie's counterexample to the statement.

The first level of coding involved the identification of the mathematical objects used by Cassie (*resources*) and the acts performed on or with them (*actions*). For instance, the inequality condition from the task statement was identified as a resource to perform the action of generating an example sequence. The second level of coding was to infer the intention behind Cassie's use of the resources and actions. In the above transcript, all the identified actions and resources seemed to be expressly used with the intention of generating an example sequence to refute the statement. Thus, the second level of coding generated *bundles* of actions and resources based on their common intention. This was done iteratively for every task worked on by every participant. The final level of coding involved comparing the bundles identified across every task for each single participant and across every participant for each single task. For more details about data collection and of the GT phases of analysis see Karunakaran (2014).

Like typical GT studies, the claims generated for the expert and novice groups in this study were solely derived and supported by the data. This was achieved by constant and repeated examinations of the data for confirming and for disconfirming evidence. Since the emergent claims during the GT process were constantly evolving based on the confirming or disconfirming evidence found with continuing analyses, the final claims put forth by this method were necessarily consistent with the corpus of data collected within each group.

Knowledge in Pieces (KiP) and Knowledge Analysis (KA)

KiP models knowledge as a system of diverse elements and complex connections. One of the main principles of KiP is that knowledge is context sensitive (Smith, diSessa and Roschelle, 1993). This means that the productivity of a piece of knowledge is highly influenced by the context in which it is used. In contrast to studies that focus on identifying students' misconceptions, KiP focuses on the ways that students build new knowledge onto their prior knowledge. Adopting this theoretical framework implies that one analysis in this paper focuses on ways that Cassie builds on her prior ideas while suspending judgment about her correctness.

There are different types of KA studies. *Microanalytic* studies are a type of KA studies, which focuses on identifying knowledge elements/resources and how they are used in real–time reasoning. Knowledge resources considered in KA do not have to be mathematical, and can be intuitive in nature. KA tends to focus on a short segment of thinking, and document moment–by–moment changes in the process by which different ideas develop in students' engagement with the topic. The KA in this paper segmented Cassie's engagement with the task into thematic episodes. It exploited any relevant data (e.g., gestures, other parts of transcripts) to optimally understand the activation of knowledge resources in various contexts. The analysis then generated multiple models (interpretations) of Cassie's argument in each episode, and put these models in competition with one another. This process of *competitive argumentation* (VanLehn, Brown, & Greeno, 1984) was used to refine interpretations of Cassie's thinking. In this paper, we illustrate the use of the counter models with the first episode of Cassie's engagement with task 1. For the rest of the episodes we only present the final model from the analysis due to space constraint.

Analysis

Grounded Theory Analysis

The GT analysis found, through comparison of bundles across the tasks and students, that if the novice provers (NPs) searched for a counterexample to invalidate a given task statement then they seemed to require an earlier rationale (intuitive, inductive, deductive, or otherwise) for the invalidity of the statement. That is, the NPs needed to already believe that a statement is invalid, before they searched for a counterexample. An alternative to this would be the strategy of using the search for a counterexample as an investigative tool, without any earlier rationale for the invalidity of the statement. This alternative strategy was more descriptive of the expert provers in the larger study.

For instance, when given the statement in Task 1 (see Figure 1), Cassie stated that, "with sequences I usually start just by like counting numbers and then seeing if it seems like it's gonna converge or not." She then assigned the first term of the sequence to be 1. Cassie went on to realize that since the inequality condition presented in the task statement did not place a strict upper bound on the terms of the sequence. She justified this by stating, "the only constraint is like in the future so a_{2n} and a_{2n+1} can grow arbitrarily large and like all that matters is that the next two, like the next a_{2n} and a_{2n+1} are bigger than that." She then made her initial conclusion that the statement of Task 1 was invalid. Cassie then explicitly expressed her intention of finding a counterexample. She proceeded to generate the sequence $a_n = \{n\}_{n=1}^{\infty}$ (where *n* is a positive integer) as a valid counterexample to the statement of Task 1. That is, the sequence $a_n = \{n\}_{n=1}^{\infty}$ satisfies the inequality condition $0 < a_n \leq a_{2n} + a_{2n+1}$, but the corresponding series diverges.

Cassie also routinely used examples that were generated only by using the constraints or assumptions present within the task statements. These types of examples are termed as *constructed examples*. For instance, when Cassie contended with a modified version of the statement in Task 1, she used only the constraints within the task statement to generate more examples. The modified task statement was identical to the original statement, except for the last word changed from "converges" to "diverges." Cassie seemed to focus on constructing an example of a sequence that expressly satisfied the inequality condition $0 < a_n \leq a_{2n} + a_{2n+1}$. She successfully did this by generating the sequence $\{1, ½, ½, ¼, ¼, ¼, ¼, ...\}$ with the goal of constructing a sequence with converging terms, and where the corresponding series diverges.

The previously described instances are exemplars of Cassie's proving behavior. This behavior was consistent with the proving behavior of other members of the greater NP group. In the next section, we present the KA of Cassie's proving process separate from her group. We analyzed Cassie's engagement with Task 1, which was split into three episodes. The analysis was organized chronologically.

Knowledge Analysis

In episode 1, Cassie came up with the sequence $\{n\}_{n=1}^{\infty}$ as a counterexample for the statement in Task 1. She relied on a transitive property of inequalities and inferred the strictness of the problem's constraint.

> *Cassie:* So if a_n were 1, uh a_1 were 1 then that would have to be less than or equal to a_2, a_3. So and then that would have to be, that would have to be less than or equal to a_4 plus a_5 plus a_6 plus a_7, a triangle inequality.

Cassie started by plugging in numbers for *n* and described the nature of the sequence based on the inequality, $a_n \leq a_{2n} + a_{2n+1}$. She immediately saw that if the terms of the sequence satisfied the inequality for all integers *n*, then $a_1 \leq a_2 + a_3$ meant that $a_1 \leq a_2 + a_3 \leq a_4 + a_5 + a_6 + a_7$. She called this a triangle inequality. We posit that she mistakenly referred to the transitivity property of inequality as a triangle inequality.

Cassie: I'm inclined to say that it's false because the, I mean the only constraint is in the future. So the a_{2n}, a_{2n+1} can grow arbitrarily large and all that matters is that the next two, like the next a_{2n} and a_{2n+1} are bigger than that. So. /…/ So like a_n is only constrained by a_{2n} and a_{2n+1}, but nothing before it. So my initial inclination is to say that it's false.

At this point Cassie saw that each term of the sequence could grow to become arbitrarily large insofar as it was smaller than its associated $a_{2n} + a_{2n+1}$. Her use of the phrase "in the future," "all that matters," and "nothing before" suggests that Cassie did not see the constraint as particularly strict, or as immediately affecting the terms of the sequence. The inequality only required that the sum of the future terms of the sequence had to be larger than the current term. Thus, given that the sequence could grow to become arbitrarily large, and there was not an immediate constraint on the rate of growth, Cassie posited that a sequence that satisfied the inequality could diverge, and thus the statement was false.

Cassie: So let's see (pause) um (pause) yeah I mean it, it seems like the sequence just a_n equals n wouldn't converge because obviously we have the n's less than or equal to $2n$ plus $2n+1$. So that meets the criteria but /.../ which means the a_n's are going to infinity. So the sum wouldn't converge.

Cassie used the sequence a_n as a counterexample to disprove the claim in task 1.

Model. By plugging in values for n, Cassie immediately recognized the behavior of the terms of the series. Applying the transitive property, she concluded that $a_1 \leq a_2 + a_3 \leq a_4 + a_5 + a_6 + a_7 \leq \cdots$. This seemingly "nested" quality of the terms did not play a major role in the process of coming up with the counterexample. Cassie asserted that the terms of the sequence could grow to become arbitrarily large, and she deemed the constraint of the terms' needing to be smaller than a sum of two consecutive future terms to be relevant, but not immediately consequential to the growth of the terms. With this she came up with the sequence $a_n = \{n\}_{n=1}^{\infty}$ as a counterexample to disprove the claim in Task 1.

Counter Model. Cassie initially considered the sequence $a_n = 1$ ("if a_1 were 1,"), which she would return to later. However, she noticed that the terms were only constrained by a_{2n} and a_{2n+1}, but "nothing before it", and so any a_i's before a_{2n} could behave without constraint. Thus, the unpredictability of the terms before a_{2n} and a_{2n+1} led her to believe that it could diverge. So she came up with the $a_n = \{n\}_{n=1}^{\infty}$ as a counterexample because it satisfied her initial thought of $a_1 = 1$ and a_n grew to become really large before it reached a_{2n}.

We posit that students' taking $a_1 = 1$ is a common practice. After that, Cassie worked with an arbitrary sequence a_n, and did not revisit $a_1 = 1$. She later came up with the sequence $a_n = 1$, but only after she recognized the lack of strict inequality in the constraint. The counter model suggests that Cassie did not treat the sum of a_{2n} and a_{2n+1} as the bound, but instead the individual terms. Her acknowledgement that $n \leq 2n + 2n + 1$ with her counterexample refutes the counter model. Cassie understood that the inequality constrained the terms, but attributed the looseness of the constraint to the possibility of the terms diverging, which motivated her counterexample.

In episode 2, Cassie constructed an example of a sequence that satisfied the inequality, and where the corresponding series diverges, by recognizing and utilizing the lack of strict inequality in $a_n \leq a_{2n} + a_{2n+1}$. Using the transitive property from earlier, she constructed a sequence $\{a_n\}$ where $a_1 = 1$, $a_2 = a_3 = ½$, and $a_4 = a_5 = a_6 = a_7 = ¼$ and so on, such that $1 = a_1 = a_2 + a_3 = a_4 + a_5 + a_6 + a_7$. She generalized the pattern associated with this example, where the 1's were made of 2^{n-1} many terms of $(½)^{n-1}$. She effectively used a sequence $b_n = 1$ as a counterexample, where b_n followed the pattern $b_1 = a_1$, $b_2 = a_2 + a_3$, $b_3 = a_3 + a_4 + a_5 + a_6$, etc. Cassie asserted that her example was a defining example for the claim that any series satisfying the task's criteria must always diverge.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

In episode 3, Cassie justified that claim by arguing that it was not possible for any series that satisfied the given criteria to be convergent. Cassie explained that with two general related sequences (say, a_n and b_n), if b_n was a strictly increasing sequence, then $\sum b_n$ (and thus, $\sum a_n$) would be infinity (The Divergence Theorem), and therefore the series would not be convergent. Cassie asserted that the best–case scenario for the series to be convergent would be if all the b_n's were equal, like with her counter example. Cassie proved the statement by using her example as a boundary case.

In summary, Cassie's engagement with the task and the modified task is sophisticated. She productively inferred the implication of the loose constraint of the inequality to the rate of growth of the sequence, and used it to find the first counterexample. She attended to relevant details about the task, like the lack of strict inequality to construct her example. She recognized $\sum 1$ as an example, and using the transitive property constructed and generalized the sequence 1,½,½,¼,¼,¼,¼, …. She recognized her example as a boundary case and worked with two different, albeit related sequences, a_n and b_n to prove the modified statement.

Discussion and Implication

The results from the two analyses provide insights into Cassie's proving process, albeit at different grain sizes. The Grounded theory (GT) analysis provided broader patterns about Cassie's process as a representative of the novice prover (NP) group across the five tasks. Particularly pertaining to the need to find a counterexample to disprove a statement, Cassie needed prior conviction (grounded in a particular rationale) about the fallacy of the statement. To then construct it, Cassie solely focused on the explicit constraints set in the problem statement.

Knowledge Analysis (KA) focused on the details of Cassie's engagement with one task to identify knowledge resources that were influential in her proofs. While the GT analysis also started with identifying actions and resources, albeit strictly mathematical, the ultimate result of that analysis necessarily removed details that do not apply across the NP population. KA remained at that level of detail in providing specific insights about Cassie's proving process.

KA also highlights the power of Cassie's productive inferences in constructing a counterexample, and to generalize from an example. Cassie did need the prior conviction before constructing a counterexample. However, that prior conviction came from a productive inference she made about the loose constraints of the inequality on the growth rate of the terms of the sequence. In addition to providing the prior conviction, her attention to task constraints (lack of strict inequality) also allowed her to construct a counter example ($b_n = 1$). She then productively inferred that a sequence satisfying the equation, $a_n = a_{2n} + a_{2n+1}$, was a boundary case, and used her example to prove the modified claim. KA confirms the findings from the GT analysis, but allows us to observe Cassie's sophistication through the details of her proving process.

These two analyses together provide a more complete picture of Cassie's proving process. On the one hand, this is common sense. Employing complementary methods results in richer analysis. On the other hand, to our knowledge this might be one of the first studies that uses GT analysis in concert with KA, thereby crossing the boundaries between the two methods set out in Parnafes & diSessa (2012). We now discuss the importance of considering these two methods together, and the danger of privileging only one of the analyses.

Combining the two methods provides a more accurate positioning of Cassie as a prover. The result of the GT analysis positioned Cassie as a consistent member of the novice prover group, whereas KA was able to uncover Cassie's sophistication. Adiredja (2015) has argued that theoretical perspectives of cognition hold the power in determining *what* counts as productive mathematical practice, and who are deemed as "successful" learners, highlighting the connection between cognition and equity issues. While KA and KiP value and prioritize knowledge and sense making of novice learners, and are against treating novices' knowledge as a subset of that of experts, GT is not tied to any such particular theoretical perspectives. In fact, the GT analysis in this paper was

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

particularly mindful of the danger of positioning the novice group in a deficit way. However, by grouping students a priori as novices and experts, the study was in danger of beginning with a particular positioning of students before the analysis even started. This discussion highlights the importance of framing for GT studies, and how framing, in addition to theoretical perspectives also contribute to students' positioning. We were able to mediate that concern by combining the two methods.

Combining the two methods also mediates some generalizability concerns of KA and problematizes the common novice–expert dichotomy. By favoring depth and richness of analysis of cognition, one of the potential limitations of the findings from the KA done in this paper is the lack of immediate generalizability of its findings to other subjects. In a typical microanalytical KA study, the analysis would continue to identify particular kinds of resources or knowledge elements that Cassie used. Then it would examine the generalizability of those theoretical entities with other students. As is, in this paper, while KA was able to show Cassie's sophistication in her proving process, that sophistication is unique to Cassie. Little can be said about how undergraduates generally prove or construct counterexamples.

The GT analysis grounds Cassie's sophistication in her membership in the NP group as implicated by her proving process. Without the GT analysis, any resemblance of Cassie's proving process to those of an expert, or any sophistication could be attributed to her being an exception. The fact that Cassie's proving processes were consistent with those of the NP group, allows her sophistication to challenge the novice–expert dichotomy. Researchers have argued against the over–privileging of experts' knowledge, and suggested the shift in focus to understanding novices' knowledge in their own terms (diSessa et al., 2016; Smith, diSessa & Roschelle, 1993). In fact, Weber et al. (2014) have shown the continuity between novice and experts proving behaviors.

In summary, putting the two methods in communication with each other proved productive. The varying grain sizes in the results of the analyses provide a more complete reporting of the patterns we observed about Cassie's proving process. Mindful of the role of cognitive studies in positioning students, the two methods together also provide a more accurate positioning Cassie as a student. Related to that point, we problematize the common expert–novice framing of studies. Our dual analyses highlight how framing, independent from methods and theory, also contribute to the positioning of students. At the same time, the analyses also highlight the power in combining the two methods in challenging the subset model of novices' knowledge to that of experts.' By only focusing on Cassie, we were able to see the power of her proving process and the important inferences she was able to make.

Acknowledgments

Both authors contributed equally to this work.

References

Adiredja, A. P. (2015). Exploring roles of cognitive studies in equity: a case for knowledge in pieces. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1269–1276). East Lansing, MI: Michigan State University.

Alcock, L. & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics*, *69*(2), 111–129.

Charmaz, K. (2000). Grounded theory: Objectivist and constructivist methods. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 509–535). Thousand Oaks, CA: SAGE.

Charmaz, K. (2001). Grounded theory. In R. M. Emerson (Ed.), *Contemporary field research: Perspectives and formulations* (2nd ed., pp. 335–352). Prospect Heights, IL: Waveland Press.

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative research.* Thousand Oaks, CA: SAGE Publications.

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches* (2nd ed.). Thousand Oaks, CA: Sage Publications.

diSessa, A. A., Sherin, B. & Levin, M. (2016) Knowledge Analysis: An introduction. In A.A. diSessa, M. Levin, & N. J. S. Brown. (Eds.). *Knowledge and interaction: A synthetic agenda for the learning sciences.* New York, NY: Routledge.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction*, *10* (2–3), 105–225.

Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago, IL: Aldine Publishing Company.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, *44*(1), 5–23.

Karunakaran, S. S. (2014). *Comparing bundles and associated intentions of expert and novice provers during the process of proving* (Unpublished doctoral dissertation). The Pennsylvania State University, University Park, PA.

Mejia–Ramos, J. P., & Tall, D. O. (2005). Personal and public aspects of formal proof: A theory and a single–case study. In D. Hewitt & A. Noyes (Eds.), *Proceedings of the 6th British Congress of Mathematics Education* (pp. 97–104). London, UK: BSRLM.

National Council of Teachers of Mathematics. (2000). *Principles and standards of school mathematics*. Reston, VA: NCTM.

Parnafes, O., & Disessa, A. A. (2013). Microgenetic learning analysis: A methodology for studying knowledge in transition. *Human Development*, *56*(1), 5–37.

Raman, M. J. (2002). *Proof and justification in collegiate calculus* (Unpublished doctoral dissertation). University of California, Berkeley, Berkeley, CA.

Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioural schemas and the proving process. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 199–215.

Smith, J. P., diSessa, A. A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *Journal of the Learning Sciences*, *3*(2), 115–163.

Strauss, J., & Corbin, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.) [Kindle edition]. Los Angeles, CA: Sage Publications.

VanLehn, K., Brown, J.S., & Greeno, J. (1982). Competitive argumentation in computational theories of cognition (Xerox Cognitive and Instructional Science Series Report No. CIS–14). Palo Alto, CA: Xerox.

Weber, K. (2001). Student difficulty in constructing proof: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119.

Weber, K., Inglis, M., & Mejia–Ramos, J. P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and research on epistemic cognition. *Educational Psychologist*, *49*(1), 36–58.