# TRANSCENDING TRADITIONAL/REFORM DICHOTOMIES IN MATHEMATICS EDUCATION

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*Here, we report on the development of a theoretical framework for mathematics teaching that looks aside from common traditional / reform dichotomies. Attention is focused on managing the amount of new information students must attend to, while structuring that information in a manner that allows discernment of key features and continuous extension of meaning. Drawing on and combining ideas based on mastery learning and on the variation theory of learning, we propose an alternative where fluency and emergent knowing are inseparable and mutually reinforcing, and motivation is based on success and the challenge of pressing the boundaries of knowing. Gains in student achievement associated with this framework have been significantly faster than national norms.*

Keywords: Mathematical Knowledge for Teaching, Design Experiments, Instructional Activities and Practices, Elementary School Education

## Introduction and Purpose

Recent calls to find a balance between allegedly opposite instructional approaches—commonly dichotomized as "traditional" and "reform" approaches—have at times posited these extreme approaches as complementary, suggesting that the two may productively interact. For instance, Ansari (2015) claimed that "[I]t is time to heed the empirical evidence coming from multiple scientific disciplines that clearly shows that math instruction is effective when different approaches are combined in developmentally appropriate ways" (para. 14). In this paper, we argue that such seemingly contradictory approaches are in fact very similar in significant ways and propose a 'third way' that addresses important features not stressed by either approach.

The work reported here developed from our work with the Math Minds initiative, which is a five-year partnership between a large school district, a mathematical charity, a children's support group, a university education faculty, and a funder. It is aimed at improving mathematics teaching and learning at the elementary level. More specifically, it aims to deepen understanding of relationships between teachers' knowledge, curricular resources, professional development, and students' performance, a combination not commonly addressed in the literature. In particular, we are working to identify features of resources that can support the development of teachers' mathematical knowledge for teaching. Here, we outline key principles that we have identified as significant for teaching mathematics and then contrast these principles with other approaches. While our primary aim here is theoretical, we also provide a brief statement regarding impact on student achievement.

## Theoretical Perspectives

While there are widely varying approaches to teaching mathematics, most approaches may be placed somewhere along a continuum with respect to the degree of emphasis they place on (A) mastery of fixed algorithms as a means of achieving procedural fluency and (B) conceptual understanding and mathematical process (*cf*. Star, 2005). They may similarly be placed along a second continuum according to the degree of (A) teacher guidance or (B) student responsibility for developing their own strategies and procedures (*cf*. Chazan & Ball, 1999). While both A's are often

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associated with traditional approaches to teaching mathematics and both B's are often associated with reform approaches (*cf.* National Council of Teachers of Mathematics [NCTM], 1989), even attempts to balance (A) and (B) must by definition buy into one or both dichotomies (*cf.* Ansari, 2015; NCTM, 2006; Common Core State Standards Initiative [CCSSI], 2015). What we propose is not a compromise, but something that shifts attention to features seldom addressed by either: In this light, A's and B's emphases turn out to have some surprising similarities. Perhaps most notably, both A and B (and points between) typically require students to attend simultaneously to multiple pieces of new information.

Although the notion of cognitive load has been primarily used to critique Type (B) (*cf.* Kirschner, Sweller, & Clark, 2007), it can also lead to difficulty in Type (A) approaches that present complex procedures that require simultaneous attention to multiple pieces of new information. Here, Marton's (2015) Variation Theory of Learning provides a helpful framework for analysis. Marton argued that "we can only find a new meaning through the difference between meanings" and that "the secret of learning is to be found in the pattern of variance and invariance experienced by learners" (p. xi). More specifically, these patterns of variation are divided into three main categories: contrast, generalization (or induction), and fusion. Contrast allows separation or discernment of a critical feature by varying only the thing one wants to draw attention to; generalization separates the ways a previously discerned feature or object can vary, and fusion recombines features that have been separated. When variation is not carefully structured, learners may overlook significant discernments critical to the intended object of learning. In addition, individual items may be perceived as either unique and difficult or as boring and repetitive, with no further meaning to be potentially gleaned from the juxtaposition of different items or from their combination in increasingly complex arrangements.

Type (B) approaches typically aim to engage students in mathematical contexts that help them make sense of mathematical relationships through processes such as problem solving, reasoning, proof, communication, representation, and making connections (*cf.* CCSSI, 2015; Kilpatrick, Swafford, & Findell, 2001; NCTM, 2000; Western and Northern Canadian Protocol, 2006). While it is possible to attend to these features in conjunction with mastery and careful variation, the importance of doing so is typically not made explicit. For example, in our own work outside of this project, we have noted that with an emphasis on multiple strategies, students have often remained unaware of the connections between them. Problems too complex for students to unpack on their own have prompted teachers to do so *for* them. Further, some students have engaged in ways that allowed them to bypass key learning objectives.

In documenting the difficulties teachers sometimes face when transitioning away from transmission-based models of math instruction (Type A), Swan, Peadman, Doorman, and Mooldjik (2013) noted that teachers may "at first withdraw support from students and then recognise the need to redefine their own role in the classroom" (p. 951). In our work, we have not asked teachers to begin with complex problems nor to withdraw key supports but have instead focused directly on what *different* supports might look like, and how these might contribute to both sense-making and continuous extension of mathematical knowing.

A key motivating principle of this study is the conviction that virtually all students are capable of learning challenging material (in our case, mathematics) if given appropriate supports (Bruner, 1960). From the outset, we discussed with teachers the importance of nurturing a growth mindset (Blackwell, Trzesniewski, & Dweck, 2007), which includes the belief that mathematical ability is learnable rather than innate. We explicitly advocated a mastery approach to learning (Guskey, 2010), with an emphasis on parsing material into manageable pieces, assessing continuously (also see Wiliam, 2011), moving forward as students demonstrate independent mastery, and extending as needed to ensure challenge for all. We further distinguish our approach through use of what we call *micro*-variation, in which we treat variations that might otherwise be seen as trivial as legitimate

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obstacles (Metz et al., 2015). By structuring variation in a responsive manner, we have worked to nurture classroom environments where all can succeed and be challenged. Motivation is thus based on success and continuous growth (Malone & Lepper, 1987; Pink, 2007).

Building on the Marton's Variation Theory of Learning (*cf.* Gu, Huang, & Marton, 2004; Kullberg, Runesson, & Mårtensson, 2014; Marton, 2015; Park, 2006; Runesson, 2005; 2006; Watson & Mason, 2005; 2006), we have worked to avoid the fragmentation that can happen when curricula are parsed into tiny pieces. By using systematic variation (Park, 2006) to draw attention to key ideas that are often overlooked and by considering how these may change within and eventually between topics, we have aimed to support the development of rich webs of interconnected understanding (for both teachers and students). These webs then allow the continued emergence of new understanding (Davis & Renert, 2014).

## Mode of Inquiry

We began the study with the intent of exploring the impact of a mastery approach to learning with an explicit focus on structured variation. The study draws on multiple sources of data (classroom observation, video-taped classes, teacher and student interviews, informal conversations with teachers, and standardized tests) to inform next steps. The principles that currently guide our work have been both informed by and used to inform a supporting resource and an associated professional development program for participant teachers. The primary study site is a small K-6 elementary school with a history of low achievement. Our work is consistent with Cobb, Confrey, diSessa, Lehrer, and Schauble's (2003) description of design-based research in that it involves "both 'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (p. 9). Also consistent with a design focus, our work has been subject to ongoing test and revision, continuously informed by classroom observations and interactions with teachers and students. It has also developed in response to insights gained through regular meetings among members of the research team and meetings involving the research team, school leaders, and a representative from the teaching resource used to support the initiative. Relationships among initiative partners are deeply reciprocal: School leaders contribute their expertise and inform the other partners of school-based needs as well as learn from the expertise of the others; the person who represents the resource offers support to both teachers and to the research team in accessing key features of the resource and maximizing their potential, while also gathering feedback that will be used to improve the resource. The research team relies on the insights and expertise of the school-based leaders and the resource representative, while offering feedback based on research observations.

As part of the initiative, teachers were provided with guides, resource materials, and student materials that support our emphases on growth mindset, mastery learning, continuous assessment, and careful attention to variation. With these in place, ongoing professional development has emphasized the importance of appropriate parsing and pacing of instruction, continuous assessment and feedback, and strategies for extending work beyond that provided in the resources. In working with teachers as they attempt to use these ideas and materials, we have found it useful to clarify distinctions between this approach and the more familiar (A) and (B) approaches described earlier as dichotomies; while perhaps overgeneralized if taken all together and taken in their extreme forms, these have provided useful points of contrast for what we here describe as approach (C).

## Distinguishing a Third Way

We now offer a summary of our current thinking on these matters. Here, we refer to (A) and (B) to elaborate the dichotomies we presented earlier, while (C) represents our own current thinking. Importantly, we see (C) not as a balance but as something significantly different from either (A) or (B). Perhaps most notably, (C) *asks students to attend to one new idea at a time, where both (A) and*

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*(B) tend to require simultaneous attention to multiple topics*, whether those be (A) elements of complex procedures or (B) multiple topics / strategies. By systematically varying one thing at a time, (C) invites continuous extension and elaboration of meaning rather than the articulation of meaning found in given applications (as is common in A) or broad contexts (as in B). Therefore, in (C), fluency and understanding are mutually supporting rather than competing. Considerations of the role of practice follow closely from these views, with (A) using repetitive practice to build fluency, (B) using embedded practice presumably made meaningful through context, and (C) offering practice through continuous extension. Positions (A) and (B) tend to result in a wide range of achievement: In (A), some students successfully master content, while others do not, where (B) intentionally allows multiple entry points and open-ended paths. Position (C) supports the mastery of a common base of understanding that may be personalized through extension.

In working with teachers, we have found it important to draw attention to the manner in which selected resources offer variation, both so that teachers can draw student attention to such variation and so that they can adapt given examples in ways that support struggling learners and those who require extension. In Tables 2 and 3, we offer examples to clarify the distinctions we are attempting to make; here, Task Sets A, B, and C correspond to the same distinctions we have described in terms of a traditional (A) / reform (B) dichotomy and a proposed alternative (C).

In Table 1, Task Set A offers focused practice with grouping coins to make particular values, but both the values and the numbers of coins vary from item to item. As a result, there is limited value in looking for connections between individual items: The sequence does little to scaffold such relational thinking. For instance, a student fluent in calculating differences might note that moving from \$0.30 to \$0.45 with two additional coins would require an additional dime and nickel. However, a similar transition makes less sense in moving from \$0.45 to \$0.55 with three additional coins. In addition to regrouping coins, working effectively with Set B requires a systematic approach to finding *all* combinations. If both re-grouping money values and exploring combinations are new to students, this question likely varies too much at once; importantly, only students who work systematically are exposed to meaningful variation among items, while others may choose a more random approach to finding possible coin values. In Set C, only the number of coins varies, and students are not (*yet*) responsible for *creating* the variation that prompts attention to relationships between subsequent items (as in Set B).

**Table 1: Three ways to vary coin-grouping tasks**

| A | B | C |
|---|---|---|
| Using quarters, dimes, and / or nickels:<br>Make \$0.30 with 3 coins.<br>Make \$0.45 with 5 coins.<br>Make \$0.55 with 8 coins.<br>Make \$0.60 with 6 coins. | Find all the ways to make \$0.45 with quarters, nickels, and dimes. | Using quarters, dimes, and nickels:<br>Make \$0.45 with 3 coins.<br>Can you do it with 4 coins?<br>Can you do it with 5 coins?<br>Can you do it with 7 coins? |

In Table 2, the distinctions between the sets are perhaps even more pronounced. In Set A, the numbers were chosen to allow variation in the size of number and number of factors (though of course bigger numbers do not always have more factors); while this may offer practice with creating factor trees, it does little to direct attention to ways that prime factors can make particular number structures visible, as too many features change from item to item. To successfully complete Set B, students must discern many relationships between prime factors and factors, not the least of which is that a particular pattern of unique and non-unique prime factors will always yield the same number of factors (e.g. 2 x 2 x 3 has the same number of factors as 3 x 3 x 5, as both follow an a x a x b pattern); here, the onus is on students to organize their work in a manner that allows exploration of

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particular patterns of variation. If they fail to do so, the variation they experience will be largely random. By carefully controlling how much is changing, Set C1 can be used to draw attention to the fact that regardless of how a number is factored, the prime factors are always the same (which is not obvious to many students). Similarly, C2 draws attention to patterns in the structure of numbers; teachers and students alike are sometimes surprised to discover that doubling a number does not double the number of prime factors.

**Table 2: Three ways to explore prime factors**

| A | B | C1 | C2 |
|---|---|---|---|
| Make factor trees for each of the following:<br>15<br>18<br>40<br>60 | What is the smallest number with 14 divisors?<br><br>Use patterns in prime factors to help you explore this problem. | Make a factor tree for 36.<br><br>Can you do it another way?<br>Another?<br>Another?<br><br>What is the same/different about your solutions? | Make a factor tree for each of the following.<br><br>In each set, what changes from one to the next?<br><br>8, 16, 32, 64<br><br>5, 25, 125, 625<br><br>10, 100 1000, 10,000<br><br>25, 50, 75, 100 |

In sessions for professional development, we have observed that teachers who worked through these examples noted that the tasks in the (C) sets made it easier for them to recognize important ideas as well as to create their own extensions (something that most have struggled with): They, too, were prompted to think differently by this arrangement of content. In other words, when a resource offers clearly structured variation, both teachers and students engage differently with the content. This is not to say that the tasks in A and B have *no* place if properly contextualized within an appropriate sequence; rather, our key point is that *C serves a critical purpose that is often overlooked.*

While our aim here is primarily theoretical, we note that evidence based on weekly classroom observations shows students who previously struggled have become more willing to take part, and many students have become excited to keep pushing their understanding to new levels. We have also noticed a consistent improvement in mathematics basic skills. The school's total math scores on the Canadian Test of Basic Skills (Nelson, 2014) improved over a two-year period at a rate that was significantly faster than the national normed population gains [$F(2,70)=6.977$, $p=.002$]. Overall, the model indicates that the mean total math score increased from an estimated average score of 43.5 (national average=50) to that of 47.0 [$t(60.42)=3.732$, $p<.001$], with roughly equal gains over the national normed population in each year (year one = +1.7%, year 2 = +1.8%).

## Conclusions & Significance

The approach briefly described in this paper resembles current research emerging in the UK that has emphasized combining mastery learning with structured variation (*cf.* National Centre for Excellence in Teaching Mathematics, 2014; Schripp, 2015). Two features that set our work apart from these efforts are (a) the emphasis on what we have been referring to as *micro-variation* and (b) attention to intrinsic motivation by using micro-variation to continuously elaborate understanding

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and increase levels of challenge in ways that are accessible to all. We argue that it is not enough to combine instructional approaches; rather, it is necessary to transcend the borders of restrictive dichotomies. As we suggested in the beginning, we see (C) as an alternative to approaches that, when considered in terms of how content is structured, turn out to be more similar than typical traditional vs. reform dichotomies suggest. Based on results from the first three years of the program, we propose that combining mastery learning and structured variation in a context that attends closely to continuous assessment, intrinsic motivation, and emergent knowing holds great promise for improving student achievement in mathematics and for supporting teachers in achieving these goals.

Considering this third way opens up new avenues for research. It will be important to investigate strategies to support teachers who are using this approach. We are interested in the interaction between educational curricular material and teachers' knowledge.

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