PLOTTING POINTS: IMPLICATIONS OF "OVER AND UP" ON STUDENTS' COVARIATIONAL REASONING

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In this study I investigate Saldanha and Thompson's (1998) claim that conceptualizing a coordinate pair in the Cartesian coordinate system as a multiplicative object, a way to unite two quantities' values, supports students in conceptualizing graphs as emergent representations of how two quantities' values change together. I presented three university precalculus students with an animated task showing varying values of two quantities along the axes and asked each student to sketch a graph of how the two quantities changed together. In this paper I document the difficulty students encountered when they did not conceptualize a coordinate pair as a multiplicative object. I address why the convention of "over x and up y" inhibits students from constructing a coordinate pair as a multiplicative object and I provide recommendations for supporting students in constructing coordinate pairs as multiplicative objects.

Keywords: Cognition, Modeling, Instructional Activities and Practices

Researchers continue to provide evidence that students have difficulty interpreting and constructing graphs (e.g., Monk, 1992; Oehrtman, Carlson, & Thompson, 2008). Specifically, researchers suggest that students do not typically think about graphs as representations of how two quantities' values change together (e.g., Dubinsky & Wilson, 2013; Thompson, 1994). Instead, as Moore and Thompson (2015) described, many students conceptualize graphs as shapes and curves and reason based on their perception of the shape of the graph. Moore and Thompson called this *static shape thinking* and explained that a student who engages in static shape thinking might, for example, understand slope as the property of the line that determines whether the line falls or rises as it goes from left to right.

An alternative way of thinking about graphs is what Moore and Thompson (2015) called *emergent shape thinking*. They explained,

> Emergent shape thinking involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation). As opposed to assimilating a graph as a static object, emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities. (p. 4)

Central to this conception of graphical representations is an understanding that a point in the Cartesian coordinate system represents the projections of two quantities' values whose measures are represented on the axes (Figure 1). This intersection point in the plane is the object the student then imagines tracing while engaging in emergent shape thinking.

In this paper I extend Moore & Thompson's work by examining how a student's conceptualization of points in the Cartesian coordinate system might inhibit or support her in engaging in emergent shape thinking. In particular, I address why the conventional activity of plotting points by going over *x* units and up *y* units does not support students in engaging in emergent shape thinking.

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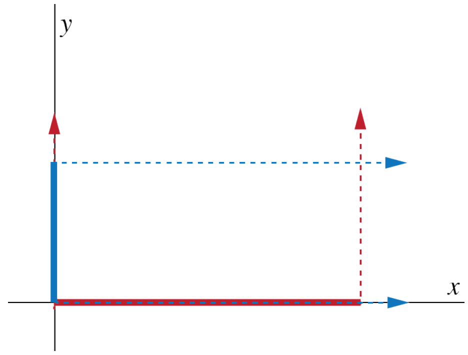

Figure 1. A point as a projection of two quantities' values represented on the axes.

Background

When one engages in emergent shape thinking she is engaging in covariational reasoning; activities involved in reasoning about how two varying quantities change in relation to each. Saldanha and Thompson (1998) provided one conception of covariational reasoning[1]. They explained,

> Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (Saldanha & Thompson, 1998, pp. 1-2)

This suggests that for Saldanha and Thompson engaging in covariational reasoning involves three mental actions: (1) conceptualizing quantities, (2) imagining quantities' values varying simultaneously, and (3) coupling two quantities through a multiplicative object.

For Saldanha and Thompson, multiplicative objects do not necessarily involve the numerical operation of multiplication. Instead, Thompson and Saldanha extend the work of Inhelder and Piaget (1964) and conceptualize multiplicative objects as mental constructions an individual makes when uniting two or more attributes simultaneously (Thompson, 2011b). Thompson provided the following examples of multiplicative objects:

1. A student can construct a rectangle's area as a multiplicative object that unites the rectangle's length and width.
2. A student can construct a point in the Cartesian plane as a multiplicative object that unites the distance of the point from the horizontal axis with the distance of the point from the vertical axis. (*ibid*, p. 47)

As Saldanha and Thompson (1998) explained, when a student constructs a multiplicative object he organizes his thoughts about the relationship between two quantities' varying values. As a result, whenever he imagines variation of one quantity he necessarily imagines the other quantity also having a value. For example, suppose a student imagines the values of x and y varying together. If the student constructs the point (x, y) in the Cartesian coordinate system as a multiplicative object then as he imagines the value of x varying he understands that the value of y necessarily has a value as well. With this conception of a point in the plane, the student can conceptualize graphs as emergent representations of how quantities' values change together.

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Methods

I conducted one-on-one task-based interviews with three university precalculus students, Sara, Carly, and Vince. All three students were in their first year of university and had declared a major in a STEM field. Thus, these students were taking precalculus to fulfill a pre-requisite for a required calculus course. The interview consisted of two phases. The first phase was a clinical interview (Clement, 2000). I engaged the students in tasks I anticipated would support me in understanding their meanings for tabular and graphical representations. The second phase of the interview was a task-based-teaching interview (Steffe & Thompson, 2000). My primary teaching goal was to support students in conceptualizing a graph as an emergent representation of how two quantities change together.

The main task in the teaching interview was based on an item from a diagnostic instrument (Thompson, 2011a). This animated item was originally designed to support researchers in better understanding in-service secondary mathematics teachers' meanings for covariational reasoning. For the purpose of this interview, I intended for this task to help me understand the nature of the multiplicative object a student constructs when she engages in covariational reasoning.

I showed each student a video that depicted a red bar along the horizontal axis and a blue bar along the vertical axis. As the video played, the lengths of the bars varied simultaneously in such a way that each bar had one end fixed at the origin. (See Figure 2 for selected screenshots from the video). The horizontal (red) bar's unfixed end varied at a steady pace from left to right while the vertical (blue) bar's unfixed end varied unsystematically. I explained to each student that the length of the red bar represented the varying value of Quantity A and the length of the blue bar represented the varying value of Quantity B. I asked each student to sketch a graph that represented how the lengths of the two bars changed together. I anticipated that students would be successful sketching a graph if they could imagine placing a point in the plane as a way to simultaneously represent the values of two quantities. Thus, this task assessed whether students saw the conventions of graphing in the Cartesian coordinate system as a way to simultaneously represent two quantities values.

The video played repeatedly until the student completed the task. The student had the opportunity to pause the video at any point. While students chose to pause videos used in previous tasks of the interview, none of the students chose to pause this video while completing the task. I engaged each student in three versions of this task. From the student's perspective, in each version of this task the length of the red and blue bars varied in different ways with respect to experiential time. From my perspective, this meant that each version of the task represented a different continuous functional relationship between the varying values of two quantities.

After I completed the interview process I engaged in retrospective analysis by transcribing each of the teaching sessions. While watching the videos and reviewing the transcriptions I identified instances that provided insights into the students' conceptualizations of graphs, points in the Cartesian coordinate system, and variation of a quantity's value. I used these instances to generate tentative models of each student's thinking that I then tested by searching for instances that confirmed or contradicted my tentative model. When I found evidence that contradicted my tentative model, I developed a new model that accounted for the student's mathematical activity. With this new model in mind, I reviewed all of the student's mathematical activity to either modify my previously constructed hypotheses or to document shifts in the student's ways of thinking.

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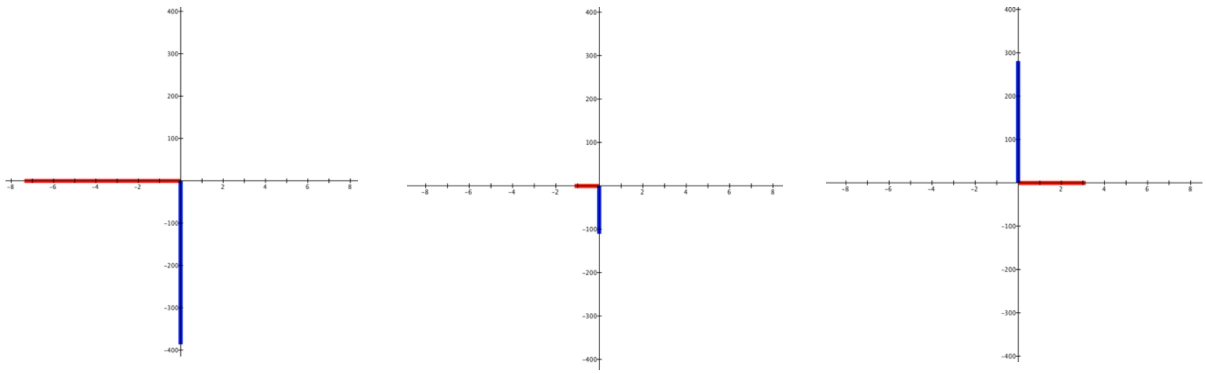

Figure 2. Three screenshots from animated task where values of Quantity A and B are represented along the horizontal and vertical axes with red and blue bars respectively.

Results

After reviewing videos and transcripts of each interview I found that while all three students plotted points from a table of values and interpreted the meaning of a point on a graph in a contextual situation, only one student independently constructed a point as a way to simultaneously represent two quantities' values when the values were represented on the axes.

At the beginning of each interview I asked the student to complete a set of conventional graphing tasks such as graphing a relationship described by a table of values and interpreting a point on his/her graph in terms of a contextual situation. Each student appropriately plotted points according to the conventions of the Cartesian coordinate system by going "over x and up y". For example, Carly explained that she plotted the point (4, -1) by going "over 4 on the horizontal and then down 1." When given a conceptual situation each student interpreted the meaning of the point in the plane in terms of the contextual situation. For example, Sara explained the point (1, 2) represented "at 1 second Susie was like 2 feet from her house."

The last task in the interview was the animated item I described above; I presented each student with a video where the length of the red bar along the horizontal axis represented the varying value of Quantity A and the length of the blue bar along the vertical axis represented the varying value of Quantity B. As the video played the lengths of the red and blue bars changed together so that each bar had one end fixed at the origin. While there were numerical values labeled along the axes, I anticipated that students would reason about the magnitude of each bar and not attend to the associated numerical values.

I presented each student with three versions of this task. I will focus on each student's first attempt at this type of problem in order to understand how students come to conceptualize a point in the Cartesian coordinate system as a multiplicative object. Figure 3 shows an accurate graph for the first version of this task.

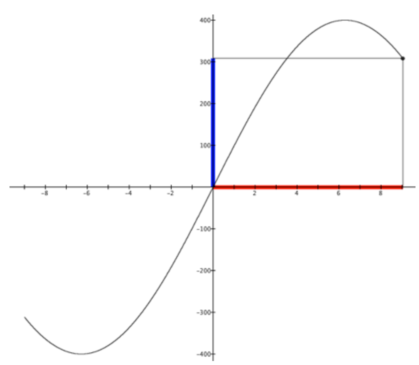

Figure 3. Accurate graph for animated item presented in Figure 2.

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Sara

As Sara watched the video, she appropriately described how the two quantities' magnitudes changed together. She explained, "As x was increasing at the beginning y was decreasing. But as x comes closer to 0 y also approaches 0 and they both increase for a little bit and as y keeps increasing or as x keeps increasing y starts to decrease." Although Sara described how the quantities' magnitudes changed together, she struggled to represent this graphically. In the following excerpt Sara explained her approach to constructing a graph from the video.

Sara: In my head I like know like as that one is increasing you have to like. I try and like think of the shape of the line or the point or whatever to get to the line. Or yeah.
Int: What do you mean you think of the shape to get to the line?
Sara: So like for this like I have to see how like. Since that [value of x] is like increasing (*gestures left to right*) and that [value of y] is decreasing (*gestures up and down*) like what I am thinking in my head. Like I am like trying to figure out which way it needs to go.
Int: Which way what needs to go? The graph?
Sara: I don't know. That is why it takes me so long when I am just staring at the graphs.

Sara appeared to abandon her thinking about changing quantities when constructing a graph. Instead of conceptualizing the graph as a trace of how the value of x and y change together, she broke the graph up into chunks based on whether the value of y increased or decreased. Then she determined a shape that depicted the appropriate behavior of y as x increased. For example, if the value of y increased as the value of x increased then she knew the graph had to go up and to the right. While Sara appropriately described how the values of x and y changed together, her tendency to engage in static shape thinking prevented her from leveraging her reasoning about how the two quantities were changing together to construct a graph.

Carly

After Carly watched the video she sketched a graph by moving her pen up and down the vertical axis and then left to right on the horizontal axis (Figure 4). Carly was trying represent the dynamic nature of the video but in the moment of acting she did not construct a way to unite the variation of the two quantities' values. When explaining her graph Carly said, "All the x values are at zero and all the y values are at zero." It seemed Carly was attending to the red bar being on the axis and not the length of the red bar. To confirm this hypothesis, I paused the video and asked Carly to determine the value of x at that moment. She told me the value of x was 0 because the red bar was on the axis. When I asked Carly to consider the length of the red bar she determined the length of the red bar was 8 and independently concluded that meant the value of x was 8. With the video still paused, she reasoned similarly about the value of y and determined the point (8, 350) represented those two values.

Attending to the length of the red and blue bar represented a critical shift in Carly's thinking. After I supported Carly in conceptualizing the length of each bar Carly described, "plotting all the points" and sketched an appropriate graph of the behavior represented in the video.

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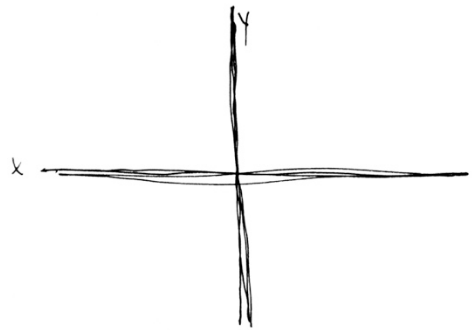

Figure 4. Carly's initial graph representing the behavior of two quantities' varying values.

Vince

After three minutes of puzzling about the task, Vince successfully represented the behavior in the video by sketching a graph in the plane. When explaining his approach, Vince described a point that he imagined as the "intersection" of the red and the blue bars and he described keeping track of this intersection as the video played. Although Vince described this approach when he first viewed the video, it took more than three minutes of reasoning for him to believe that this activity would represent how the lengths of the red and blue bars changed together.

I conjecture that Vince was able to construct the intersection point in the moment of acting because he had constructed a multiplicative object in thought that united the quantities' values. Throughout the entire interview when Vince talked about a value of x he also talked about the associated value of y. This was true whether Vince was referencing a table, graph, or contextual situation. This suggests that Vince had a way of thinking about relationships between quantities that united the values of x and y. When I presented Vince with this last task he focused on developing a way to graphically unite the values of x and y. He constructed a point as a multiplicative object in order to satisfy his need to unite the values of x and y. After conceptualizing the point as an "intersection" – a multiplicative object, he was able to represent the behavior in the video as a graph in the Cartesian coordinate system.

Discussion

All three students successfully plotted points from a table of values and gave contextual interpretations of points on their graphs. However, only Vince – with much hesitation – constructed a graph by conceptualizing a point as a way to unite two quantities' measures. These results suggest that the years of graphing practice that students endure in grade school do not necessarily prepare students for conceptualizing values in a table as measures represented along the axes nor does it prepare students for constructing a point in the Cartesian coordinate system as a multiplicative object. In the following paragraphs I hypothesize why plotting points as "over x units and up y units" does not support students in conceptualizing a point as a multiplicative object. I also provide recommendations for how educators might support students in constructing points as multiplicative objects.

When the three students in this study plotted points from a table they enacted the activity "over x and up y". The point the student plotted was the product of this activity. It is likely that as soon as the student plotted the point he/she no longer thought about the activity that produced the point he/she constructed. For a student engaged in this type of activity, it is as if there is a place called (x, y) and the student is being asked to find it. As a result, the values of x and y are tied to a single place in space, as opposed to tied to two measures on the axes. Without conceptualizing attributes to unite, the student will never necessitate constructing a multiplicative object. Thus, it is not surprising that these students had difficulty using their meaning of points – locations in the plane – to construct a

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point in the Cartesian coordinate system as a way of uniting two quantities' values that are represented on the axes.

This suggests that educators need to support students in developing an entirely new conception of graphs that is rooted in conceptualizations of quantities' varying values being represented along the axes. Carly and Sara's work provides evidence of two difficulties students need to overcome in order to develop this conception of graphs.

When Sara first engaged with the animated task she explained how the two quantities' values changed together – she engaged in covariational reasoning. However, she abandoned this way of thinking when she went to sketch a graph and instead focused on the shapes that would represent the way the quantities' values changed together. While Sara was able to construct a correct graph, she did not construct the graph by keeping track of *how* the quantities were changing together. This suggests that conceptualizing a graph as an emergent trace of two quantities' values requires more than imagining how two quantities change together. While Sara was able to successfully complete this task, her thinking was entirely dependent upon breaking up the graph into chunks where she could appropriately determine the shape of the graph. This way of thinking is constrained to situations where Sara can imagine breaking up the behavior into graphs and limits the power of Sara's ability to represent and interpret graphs as emergent traces of covarying quantities.

Carly's difficulty stemmed from not differentiating between the length of the bar and the location of the bar on the axis. In order to construct a multiplicative object, the individual must first construct two properties to then unite. I conjecture that by focusing on the aspect of the bar that remained the same – its location on the axes – instead of attending to the aspect of the bar that varied – the length of the bar, Carly was unable to conceptualize two distinct properties. When I paused the video and asked Carly to determine the length of the bar, Carly constructed two properties - the length of the red bar and the length of the blue bar, which she could then think about uniting. While the numerical values likely helped Carly coordinate the animated task with her conception of values represented in a table, once Carly constructed the point as a way to unite two numerical values on the axes, she was able to imagine keeping track of all of the points. This suggests that Carly's thinking about points as uniting two measures on the axes was not dependent on knowing the values of the coordinates.

While Carly experienced difficulty conceptualizing two attributes to unite, Vince experienced difficulty conceptualizing the point as a way to unite two properties, namely the value of x and the value of y. This suggests that constructing a point in the Cartesian coordinate system as a multiplicative object is a nontrivial activity; conceptualizing two attributes to unite, and then conceptualizing how to unite these values graphically are both cognitively demanding activities.

By the end of the interviews, both Carly and Vince were able to construct a graph by imagining "all the points" and engaging in emergent shape thinking. Looking at the shifts in each student's thinking we see that attending to the measure of each bar's length can help students conceptualize two attributes to unite. Additionally, if a student conceptualizes relationships between quantities as a way unite values of two quantities in thought, then the student has the opportunity to construct representations of this relationship be it through tables, graphs, or formulas. Thus, educators should encourage students to unite their conception of two quantities' measures when reasoning about contextual situations, tabular representations, formulas, and graphical representations.

From a researcher's perspective, the coordinate pair (x, y) necessarily unites values of x and y in the Cartesian coordinate system. However, this study provides evidence that when plotting and interpreting points in the Cartesian coordinate system, students are not conceptualizing the coordinate pair (x, y) as a way to simultaneously represent values of two quantities. Additional research is needed to better understand how students construct coordinate pairs in the Cartesian coordinate system as multiplicative objects and how students leverage their conceptualization of coordinate

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pairs as multiplicative objects when engaging in covariational reasoning and reasoning about dynamic situations.

Endnotes

[1]See Carlson, Jacobs, Coe, Larsen, and Hsu (2002) and Confrey and Smith (1995) for other conceptions of covariational reasoning.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic vents: A framework and a study. *Journal for Research in Mathematics Edu*cation, *33*(5), 352-378. doi: 10.2307/4149958

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. Lesh & A. E. Kelly (Eds.), *Handbook of research methodologies for science and mathematics education* (pp. 341-385). Hillsdale, NJ: Erlbaum.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, *26*(1), 66-86.

Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior*, *32*(1), 83-101.

Inhelder, B., & Piaget, J. (1964). *The early growth of logic in the child* (E. A. Lunzer & D. Papert, Trans.). New York: W. W. Norton & Company, Inc.

Monk, S. (1992). Students understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*, MAA Notes (Vol. 25, pp. 175-193). Washington, DC: Mathematical Association of America.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 782-789). Pittsburgh, PA: RUME.

Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (Vol. 73, pp. 27-42). Washington, DC: Mathematical Association of America.

Saldanha, L., & Thompson, P. W. (1998). Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah, K. R. Dawkins, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298-304). Raleigh, NC: North Carolina State University.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1* (Vol. 1, pp. 21-44). Providence, RI: American Mathematical Society.

Thompson, P. W. (2011a). Project Aspire: Defining and Assessing Mathematical Knowledge for Teaching Mathematics: NSF Grant No MSP-1050595.

Thompson, P. W. (2011b). Quantitative reasoning and mathematical modeling. In L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education, WISDOMe Monographs* (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.