

WHAT DO STUDENTS ATTEND TO WHEN FIRST GRAPHING PLANES IN R^3 ?

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This paper considers what students attend to as they first encounter R^3 coordinate axes and are asked to graph $y = 3$. Graphs are critical representations in single and multivariable calculus, yet findings from research indicate that students struggle with graphing functions of more than one variable. We found that some students thought $y = 3$ in R^3 would be a line, while others thought it would be a plane. In creating their graphs, students attended to equidistance, parallelism, specific points, and the role of x and z . Students' use of these ideas was often generalised from thinking about the graphs of $y = b$ equations in R^2 . A key finding is that the students who thought the graph was a plane always attended to the z variable as free.

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Introduction

The inclusion of multivariable topics in K-12 mathematics has been proposed as a way to increase mathematical competence for all students (Ganter & Haver, 2011; Shaughnessy, 2011). Because multivariable topics share similarities with their single variable counterparts, many researchers studying student learning of multivariable topics focus on how students generalise across these contexts (e.g., Dorko & Weber, 2013; Jones & Dorko, 2015; Kabaël, 2011; Yerushalmy, 1997). Graphs are critical representations in calculus, yet students struggle with creating graphs of multivariable functions (Kabaël, 2011; Martinez-Planell & Trigueros, 2012) and finding the intersection of multivariable functions' graphs and fundamental planes (Trigueros & Martinez-Planell, 2010). Additionally, students' correct understandings about graphs in R^2 may interfere with their learning about graphs in R^3 . Some students graph $f(x,y) = x^2$ as a parabola rather than as a parabolic surface or may draw $f(x,y) = x^2 + y^2$ as a cylinder or a sphere because they are accustomed to $x^2 + y^2$ representing a circle in R^2 . These examples illustrate that as students think about the graphs of multivariable functions, they generalise the ways they think about graphs in R^2 . We sought to further explore this, with the hypothesis that learning more about what students attend to when graphing can help instructors emphasize the productive connections students see across situations and address the sorts of overgeneralisations described above. Toward that end, we focus on the following research question: *what do students attend to as they first think about graphing a particular fundamental plane ($y = 3$) in R^3 ?*

Background Literature and Theoretical Perspective

Graphing in three dimensions requires students to coordinate three quantities, as well as shift from thinking of y as a dependent variable to considering a z that is dependent on x and y . This is a difficult generalisation to make. Students may give an (x,y,z) tuple as an element of the domain or range (Kabaël, 2011) or may give the range as y values (Dorko & Weber, 2013), indicating that they have not reconceptualised y as an independent quantity, and do not necessarily think of $f(x,y)$ as an output or the height of the graph at a particular (x,y) . As another example, Martinez-Planell and Trigueros (2012) described a student who drew $f(x,y) = x^2$ as a parabola on the xz plane and insisted that although the point $(2,1,4)$ satisfied the equation, it was not on the graph. Students also have trouble determining the intersection of a surface with fundamental planes (a plane of the form $x = a$, $y = b$, $z = c$ for a constant c). Trigueros and Martinez-Planell (2010) found that students who had taken multivariable calculus knew that these were planes, but weaker students had trouble placing such planes in a set of manipulatives and drawing the planes on a 2D image of a multivariable graph.

Stronger students could place the planes, but had difficulty determining the intersection of such planes with a multivariable surface.

Martinez-Planell and Trigueros' research has been in the context of developing a set of activities to help students learn how to graph multivariable functions. They found that notation may hinder students' multivariable graphing attempts. One of Martinez-Planell and Trigueros' (2012) students drew $f(x,y) = x^2$ as a parabola on the xz plane and was unsure whether or not she was in two or three dimensions. She gave the intersection of $f(x,y) = x^2$ and $y = 1$ as "two points," which the researchers interpreted as thinking of the graph in two dimensions. The same student drew a correct three-dimensional graph for $z = x^2$. Familiar notations, such as $x^2 + y^2$ (a circle in \mathbb{R}^2) may result in students thinking that the graph of $f(x,y) = x^2 + y^2$ is a cylinder or a sphere (Martinez-Planell & Trigueros Gaisman, 2013; Trigueros & Martinez-Planell, 2010). The researchers subsequently altered the activity sets to avoid familiar notations, so it is unknown if students are able to use such notations productively. We build on these authors' work by considering how students' conceptions of single-variable functions' graphs interact with their conceptions of graphs of multivariable functions. Findings from other studies indicate that students can often successfully leverage their single-variable knowledge to make sense of multivariable topics (e.g., Dorko & Weber, 2013; Jones & Dorko, 2015; Kabaal, 2011; Yerushalmy, 1997), and we wanted to study whether this was also the case when students graph in three dimensions.

Theoretically, we consider that using knowledge from a single-variable context to make sense of a multivariable context is an example of generalisation, or "the influence of a learner's prior activities on his or her activity in novel situations" (Ellis, 2007, p. 225). We think about generalisation from an actor-oriented transfer perspective (Ellis, 2007; Lobato, 2003). This perspective privileges what students see as similar across situations, even if the similarities they perceive are not normatively correct.

Data Collection and Analysis

We interviewed 11 differential calculus students who had not yet received instruction regarding \mathbb{R}^3 . We felt this would let us observe students' initial sense-making and generalisations in real time. This paper focuses on data from two tasks: students' graphs of $y = 2$ in \mathbb{R}^2 and $y = 3$ in \mathbb{R}^3 . Before giving students the second task, we showed them an image of xyz axes and explained that the xy plane was flat with the z axis perpendicular to it. We used a tabletop (xy plane) and a pen (z axis) to show students what these axes looked like in $3D^1$. We asked follow-up questions such as "why did you draw a [line, plane] here?" We audio and video recorded the interviews and used LiveScribe pens. We transcribed the interviews and used the transcripts in data analysis.

We chose to focus on these problems because of reported difficulty students experience with multivariable functions' graphs, and also because fundamental planes can help students complete graphing and other tasks in calculus (e.g., visualising graphs; the boundaries in multiple integration). Hence it seemed valuable to explore how students might think about equations of the form $y = c$ in \mathbb{R}^3 (and we specifically chose $y = 3$).

We used the constant comparative method (Strauss & Corbin, 1998) for data analysis. We first observed that some students had drawn $y = 3$ in \mathbb{R}^3 as a line, others had drawn it as a plane, and two drew a line but then thought the graph might be a plane. We hence coded students in two categories: (1) students who drew a plane or said they were unsure whether the graph was a line or plane, and (2) students who drew a line. We then looked at the data a second time, asking how students might have arrived at their answers. Students' reasoning involved words and phrases such as parallel, equidistant, "all x points," " z can be any value," " x can be any value," "I don't think that x and z really have like any effect", "all values of x and z ", variables as "not mattering," x and z being "any value," and "no matter what x or z is." We also observed students think about specific points, such as "if you say $x = 2$ and $z = 2$, it's going to be 3." We noticed that these utterances seemed to group

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themselves into three broader, non-mutually-exclusive categories: reasoning using equidistance and parallelism, reasoning about x and z , and considering specific (x,y) , (y,z) , and (x,y,z) tuples.

Finally, we looked for patterns in how students had thought about the graphs, and whether their graphs were planes or lines. That is, we looked specifically to see if there were something common to all of the students who drew the graph as a plane, and all the students who drew the graph as a line. We noticed that the difference between the graphs seemed related to whether or not the students explicitly attended to z as a free variable.

Results

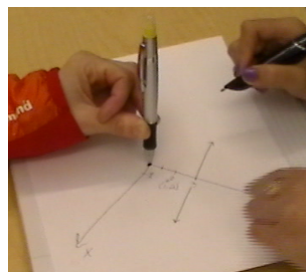
In this section, we first focus on the seven students who thought the graph was a plane or might be a plane, and then on the students who drew a line. All of the seven plane students thought about x and z values in some way. Three began by thinking about all values of x and z , then thought about specific (x,y) , (y,z) , and (x,y,z) tuples. The other four focused on all values of x and z , but did not think about specific points. We contrast these students' work with the four students who drew a line. Two of the line students thought about parallelism, one thought that the lack of x and z in the equation meant the graph would be "flattened down" and hence a line, and one seemed to have difficulty orienting himself to \mathbb{R}^3 . In this section we contrast data from the plane and line students.

Plane: Thinking about x , z , and specific points

Three students reasoned that the graph was a plane by first thinking about all x and z values, and then considering specific (x,y) , (z,y) , and/or (x,y,z) tuples (Table 1). Thinking that these points had to be on the graph of the function seemed to help S3 and S7 figure out that the graph was a plane. In particular, S3 began with two lines that looked like a plus sign, intersecting at $(0,3,0)$. She drew the horizontal line to represent that y equaled 3 no matter what x was, and the vertical line to represent that z equaled 3 no matter what y was. Testing points not on those lines such as $(x,y) = (2,3)$ and $(z,y) = (1,3)$ allowed S3 to conclude, "I guess I drew a plane." Note that while S3 thought about two variables at a time, S7 first thought about two variables at a time when she said " x evaluated at any point will be $y = 3$ " and then gave the 3-tuple $(2,3,2)$ as a point on the graph. S13, who was unsure whether the graph was a line or a plane also used a 3-tuple, commenting "So I feel like you can be given like $x = 0$, but $y = 3$, and z equals something, and I feel like that could correlate." We speculate that in thinking about specific points, the students were focused on something concrete, and this afforded their realisation that the graph was a plane. Notably, none of the students who considered specific points said that the graph was a line. This suggests that having students consider specific (x,y,z) tuples may be a powerful tool for helping them create correct graphs.

Table 1: Considering Specific Points and Drawing a Plane

S3	No matter what x or z is, y is always going to equal 3... I want to draw a line like this [indicates a line following the gridline for $y = 3$ on the xy plane] and a line like that [indicates a gridline parallel to the z axis and going through $(0,3,0)$]. So what it's saying here is if x were 1, y equals 3, or if $x = 2, y = 3$. And here too if z were 1, y is always going to equal 3 here... I guess I drew a plane... [a plane makes sense] when it is drawn out like that.
S7	So here's 3, we've got x , or the y , but then it's also going to be for all the z values, since z evaluated at any point will be $y = 3$. So I guess it would come out to be a plane... I kind of just thought since x evaluated at any point on the graph equals 3, since the function is basically saying all, it's saying $y = 3$ at all points on the graph, any point you evaluate, so if you say $z = 2$ and $x = 2$, it's going to be 3.
S13	So it would be, $y = 3$ would go, I want to say it would be like right here. [Draws a line at $y = 3$ that is parallel to the lines on the notebook; Figure 1]. I guess I'm just trying to relate it back to this, these equations [gestures to $y = 2$] and it just goes straight through... Here if it continues, it goes through x [meaning if she extends the line in Fig. 1, it will go through the x axis]. Then you would have an x and y value, which is the only reason why I feel like it's wrong. [using makeshift actual 3D axes]... I think it would go this way [Figure 2]. Just cuz it can't go this way [points to a line that would roughly go through $(0,1,0)$ and $(1,0,0)$] because then it will hit x , this way it doesn't hit any other points because z goes, negative z goes straight down. So I feel like this way it would go, it would hit the 3 ...it could be like a line or it could be like a sheet but I feel like a sheet just makes more sense because then you can do, I was just given x and y , but you can be given, I am assuming you can be given like $x y z$ and plot those points. So I feel like you can be given like $x = 0$, but $y = 3$, and z equals something, and I feel like that could correlate.

**Figure 1:** S13's first graph of $y = 3$ in \mathbb{R}^3 .**Figure 2:** S13's second graph.**Plane: Thinking about x and z (but not specific points)**

Three students determined that the graph was a plane by focusing on x and z as able to take on any value, though they did so in different ways. S8 thought about equidistance and parallelism, while S5 and S12 thought about the graph “stretching out” in the direction of a free variable(s) (Table 2).

Table 2: Thinking about x and z and Drawing a Plane

S8	<p>$Y = 3$ would be something like this, where this distance right here between each, between y and each of these axes would be 3, I think. I'm thinking that because if you take like this thing, and that would be everything except for y [shades xz plane]... I'm thinking of this plane in relation to y and having y be every distance that is 3 away from that plane. Actually, $y = 3$ wouldn't just be this circle, ... it would be an entire plane... it has to be parallel to x, and this has to be parallel to z, so it would be this plane right here that is 3 away from the plane that x and z creates... like for the last question when y is equal to 2, that is every value that is 2 away from $y = 0$, right? So I'm thinking that like $y = 0$ would be the same as this [xz plane]. So it's 3, it's 3 in the positive [y] direction it's going to be parallel to x in the same way that this line right here [draws $y = 2$ in R^2] is parallel ...to the x axis.</p>
S5	<p>So if $y = 3$, z can be any value so it extends in the z direction like that, and x can be any value, so it extends in the x direction like that, and it forms a plane on the, like that... I just thought about this sort of logic, that if there's a variable that doesn't affect the equation then it just kind of stretches out into that direction.</p>
S12	<p>So maybe, maybe it would be like this [draws and shades plane]. Well this would kind of just be like just a flat sheet of paper on the $y = 3$, because all x values are 3, and then I guess you assume that all z values, since it's only, the only variable in the equation is $y = 3$, then it would have to be $y = 3$ for all x and z values. It's kind of just like a, I think it's supposed to be like a flat sheet kind of, like a piece of paper, and it's on $y = 3$, so it's supposed to encompass all the x values for negative and positive, and all the z values for z, positive z and negative z. They're all on $y = 3$... Well, I just thought like since $y = 2$ it should be like this, so if it's $y = 3$ it's like that, like all x values are $y = 3$. And z is going this way, so it must be, since there's no z in the equation, then it must be covering all this area.</p>
S10	<p>I'm going to extend that like that, so then that's 3. And this is on that like x plane, xy plane... I mean it looks pretty linear. I mean like it might be a sheet, but... like if it weren't a line, it would definitely be a sheet that extends into the z plane... because a line is like infinitely small points, like a line of points, and then if it extended into the next dimension, it would be, it would just be another line going in the other, perpendicular to that line of points.</p>

S8 began by thinking about $y = 3$ as “three away” from the x and z axes, and drew a circle centred at $(0,3,0)$, parallel to the xz plane. Importantly, the act of drawing the circle seemed to make him realise that the x and z axes lie in the xz plane, and hence $y = 3$ would be a plane that is “every distance that is 3 away from that plane.” He appeared to confirm this idea by thinking about the plane as “parallel to x and... parallel to z ”, and referred back to the $y = 2$ question where the line was “2 away from $y = 0$ ” and parallel with the x -axis.

S5 generalised that the graph would stretch out in the direction of a free variable from having previously graphed $f(x,y) = x^2$ and noticing that because y can take on any value, the graph is a parabolic surface², which he saw as “stretching out” a parabola into a trough shape. Similarly, S12 realised that y would be 3 for all x and z values, so the graph would need to “cover” all x and z values. For these students, noticing that x and z were free allowed them to realise that the graph was a plane. S10, in contrast, did not explicitly say that z could take on any value, but her comment that the graph might be a “sheet” seemed to indicate that she knew, like S5 and S12, that z was free. Her reasoning that the graph could be a plane, however, seemed to come from finding a 3D analogue of a line. We infer this from her comment “if it extended into the next dimension.” Such an extension to

R^3 seemed to come from thinking of $y = 3$ as a line on the xy plane composed of an infinite number of points, and drawing a line through each of those points, perpendicular to the $y = 3$ line. This reasoning is somewhat similar to S3 (above) who drew two perpendicular lines, then tested points to see if the graph were a plane.

In summary, the commonality to students in the *Thinking about x and z (but not specific points)* category is that realising that z could take on any value allowed them to draw a plane. To summarize the plane students more broadly, we speculate that attending to z , whether in the form of z as a free variable taking on any values or by considering specific points, allowed these students to create correct graphs.

Line

Of the four students drew $y = 3$ as a line, two of them did not mention z . Both of these students explicitly referred back to the graph of $y = 2$ in R^2 , and generalised that since $y = 2$ is parallel to the x axis, $y = 3$ should also be parallel to the x axis. S6 described this as “following the x axis,” which seemed to us to be attending to parallelism (Table 3).

Table 3: Attending to Parallelism to Draw a Line

S6	So $y = 3$. So on an xy graph [draws R^2 axes] at 3, would be going this way. So on the y , following the x . So [switches to R^3 axes] this would be on the y , this is the 3 point on the y , and it's following the x axis.
S9	So if this is, if this is y is equal to 3, then I'm wondering if it will just go parallel with the x axis, but just at, but just at the y is equal to 3 point... just kind of like looking here [goes back to $y = 2$] it's like parallel to the x axis at whatever y point because it's just that straight line, so if this is like y is equal to 3, I'd assume just for all x points that you know y would equal to 3. [I think the graph is a line] just cuz when it's just equal to you know 3, it just means no matter what x value there's like nothing, it's just 3. So it would be a line, there's no, you know, changing or anything because your y is only 3. It's not like interacting with, with anything else.

Unlike S8 (above), these students did not mention the z variable. We believe that S8's attention to z , which came in the form of shading the xz plane, allowed him to see that $y = 3$ would also be parallel to the z axis, which in turn helped him see that $y = 3$ was a plane. If we contrast the three students who used parallelism, two of them drew a line but did not attend to z , and one drew a plane and attended to z . This suggests to us that thinking explicitly about the z variable as free supports students in drawing correct graphs.

The final two students thought about z , but rather than realizing that z is free, these students thought z and x “can't change” or that they “don't have any effect” (Table 4). S11 mentioned that the symbols z and x were absent from the equation, and she then referred back to the $y = 2$ graph and said “it'd still just be like this,” though she recognised that she was in 3D. We infer that because the $y = 3$ equation and $y = 2$ equation look the same (both having no x and no z), S11 generalised that $y = 3$ in R^3 would be a line just like it is in R^2 .

S1 had trouble orienting himself in R^3 , which may have contributed to his drawing a line. That is, he seemed to have so much trouble figuring out the structure of the three axes that he would have been unable to think of specific points, or coordinating parallelism in multiple directions. The student also assumed that, rather than being free, the z and x variables could not change at all. This may have resulted in his drawing a line contained on the xy plane, because such a line never intersects the x or z axis. Thinking about x and z as unable to change directly contrasts thinking of them as able to take on any value, as S3, S5, S7, and S12 (who all drew planes) did. This suggests that part of students' success in graphing $y = 3$ depended upon their ability to think of z as free.

Table 4: x and z as Not Changing and/or Having No Effect

S11	It stays on like that like xy plane because there's no like z and there's no x , so it'd still just be like this [points to $y = 2$ in R^2], but flattened down... because it's still just like a constant number, so no matter like, no matter like what the other ones are, it's just going to be that one number, which is just like a straight line across the thing... I don't think that x and z really have like any effect to $y = 3$ because there's no x and z variable in there, but it just like makes it like 3D... But it's, the function's still just like one line on the y axis.
S1	So the z direction, the z can't change and the x can't change, and the y would kind of have to be like at a diagonal [<i>the line $y=3$, when drawn on standard axes on paper looks to be at a diagonal</i>], but staying at $y = 3$. I can't really like see how to do it, but you'd find the spot where $y = 3$ for both $x = 0$ and [<i>the interviewer interrupted to ask the student to draw on the paper</i>] this might be a struggle, I'm not sure if I know - so this is going up, this is going this way, x , y , z ... Yeah, z is the vertical here. The x value, so yeah. I'm not really sure which one I should mark as $y = 3$, though... so this would be $y = 0$, I'll just say this is the positive direction, 1, 2, 3, so $y = 3$ here, and I guess I would just use this line for y always equal to 3.

Discussion and Conclusions

We found that when students first try to graph $y = 3$ in R^3 , some draw a line and some draw a plane. The students who drew planes seemed attend to z in a way that allowed them to realise that z is a free variable. Some students did this by thinking that z could take on any value, and hence the graph would extend in the z direction. Others tested particular points, indicating that they knew they could pick any value for z . Realising these points satisfied the $y = 3$ criteria seemed to help these students see the graph as a plane. Finally, students attended to parallelism and equidistance from the $y = 2$ case. Nearly all of the students referred back to the $y = 2$ in R^2 question, and their ways of thinking about the multivariable graph generalised properties from this single-variable case. For instance, some generalised $y = 2$ for all values of x in R^2 so y would equal 3 for all values of x and z in R^3 , and others generalised that $y = 2$ is parallel to the x axis in R^2 so $y = 3$ is parallel to the xz plane in R^3 .

That some of the students in this study drew $y = 3$ in R^3 as a line, as it would be in R^2 , supports Kabaal (2011) and Martinez-Planell and Trigueros' (2012) findings that students' knowledge of the shapes of graphs in R^2 may interfere with their learning about graphs in R^3 and Trigueros and Martinez-Planell's (2010) finding that students have difficulty with fundamental planes. However, other students in our study productively leveraged properties of $y = 2$ in R^2 to correctly graph $y = 3$ in R^3 . This suggests that instructors can build on students' intuitive notions, perhaps first eliciting what students think a graph might look like and then asking what role z plays in such a graph. Finally, because some of our students found thinking about specific points to be helpful, we agree with Martinez-Planell and Trigueros (2012) that instructors can help students think about 3D graphs by asking whether a particular (x,y,z) -tuple is on the graph. Further, we agree with Martinez-Planell and Trigueros (2012) that students should have the experience of constructing a 3D graph point-by-point. This affords learning that an $f(x,y)$ value is the function height at a point (x,y) , and we think it primes students with the strategy of thinking of specific points as being useful for determining the shapes of graphs.

The setup of the $y = 2$ in R^2 and $y = 3$ in R^3 tasks is a possible limitation of this study. One could argue that the order of the questions prompted students to refer back to the $y = 2$ task. While this is true, the fact that many students referenced back to it in productive ways (e.g., equidistance,

parallelism) implies that instructors might consider pairing graphs like $x = a$ and $y = b$ in \mathbb{R}^2 with an introduction to fundamental planes.

Endnotes

¹Such a demonstration does not guarantee that students understood how such a coordinate system works; in fact, researchers have found that students often must develop a schema for \mathbb{R}^3 through specific actions like plotting (x,y,z) -tuples and working with fundamental planes (Martinez-Planell, & Trigueros, 2012; Trigueros & Martínez-Planell, 2010).

²S5 was the only student to have graphed this multivariable function before the $y = 3$ task analysed here.

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