UNDERGRADUATE STUDENTS' PERCEPTION OF TRANSFORMATION OF SINUSOIDAL FUNCTIONS

Masomeh Jamshid Nejad
Simon Fraser University
masonmehj@sfu.ca

Trigonometry is one of the fundamental topics taught in high school and university curricula, but it is considered as one of the most challenging subjects for teaching and learning. In the current study Mason's theory of attention has been used to examine undergraduate student's perception of the transformation of sinusoidal functions. Two types of tasks – (A) Recognizing sinusoidal functions and (B) Assigning coordinates – were used in this study. The results show that undergraduate students participating in this study experienced difficulties in identifying a period of a sinusoid, especially when it was a fraction of π radians.

Keywords: Post-Secondary Education, Algebra and Algebraic Thinking

Background

Trigonometry has a long history. Ancient people used trigonometry for different purposes. For example, Egyptians applied trigonometry to determine the correlation between the lengths of the shadow of a vertical stick with the time of day. Astronomers also used trigonometry to find the longitude and latitude of stars, as well as the size and distance of the moon and sun. However, trigonometry was not an essential part of mathematics textbooks until a Persian mathematician named Khwarizmi introduced trigonometric functions to the world. Since then, trigonometry has become one of the main topics in high school and university mathematics books and students are required to assign time for learning trigonometry, especially trigonometric functions. This is the case since a strong foundation in trigonometric functions will likely strengthen their learning of various mathematical topics, such as Fourier series and integration techniques (Moor, 2010). It is also shown that understanding calculus and analysis is dependent on the learning of trigonometric functions (Hirsh, Weinhold and Nicolas, 1991; Demir, 2011). However, learning and understanding trigonometric functions is a difficult and challenging task for students, compared to other mathematics functions, such as polynomial functions, and exponential and logarithmic functions. While other functions (e.g., logarithmic functions) can be computed by performing certain arithmetic calculations expressed by an algebraic formula, trigonometric functions involve geometric, algebraic and graphical concepts and procedures, simultaneously (Weber, 2005, Demir, 2011).

Despite its importance and its complexity, research on trigonometry is sparse and quite limited. In the literature, only a small number of studies concentrate on students' learning of trigonometric concepts, and in particular trigonometric functions (e.g. Gray and Tall, 1991; Brown, 2005; Weber, 2005, Moor, 2010). Challenger (2009), Moore (2010) and Weber (2005, 2008) indicated that students often have difficulty using sine and cosine functions defined over the domain of real numbers. Thompson (2008) also noted that students are unable to construct understanding of the trigonometry of right angles and the trigonometry of periodic functions. In a study of undergraduate students, Weber (2005, 2008) agreed that students could not rationalize various properties of trigonometric functions or reasonably estimate the output values of trigonometric functions for various input values. Kendal and Stacey (1997) concluded that students had difficulty interpreting trigonometric functions in the unit circle, recognizing that x and y coordinates of a point on the unit circle are cosine and sine values of corresponding angles compared with other determined trigonometric functions in terms of a right triangle.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

In spite of all the research efforts in the area of teaching and learning trigonometry, especially trigonometric functions, there are still gaps in the literature. There is no research study that focuses on the concept of the transformation of sinusoidal functions; the current research attempts to fill this gap.

In order to deal with the transformation of sinusoidal functions, students need to understand the notion of the 'period of a function.' The period of a function is the distance (x value) in which function values repeat themselves. In the case of the canonical sine function $f(x) = sin\ x$, the period is 2π, the circumference of the unit circle. Considering the standard format for the sinusoidal function $f(x) = Asin(Bx \pm C) \pm D$, students are required to identify the relationship between the coefficient of x (B in the function) and the period when dealing with the transformation of sinusoidal functions. As such, the research questions are: How do undergraduate students identify period? How do they recognize the period on the graph of the sinusoidal functions?

Data Collection and Analysis

This study is part of a bigger project which examines undergraduate students' perception of the transformation of sinusoidal functions. In the larger study, seven undergraduate students from a large North American university participated They were selected from among students who had either completed a Calculus I course and were enrolled in Calculus II (3 students) or they were in a Calculus I course (4 students) in the Mathematics Department. Participants volunteered their time to contribute in the study right after I made a general request from all the classes (Calculus I and II). For the purpose of this research report, I focus only on the performance of one of the participants, Emma. She was studying Applied Science and was enrolled in a Calculus II course at the time of her interview.

A 60-minutes task-based interview was conducted and Emma was required to complete two types of interview tasks: **A)** Recognizing sinusoidal functions and **B)** Assigning coordinates. Both types of tasks were presented with the help of the Dynamic Geometry software, Sketchpad. For the 'Recognizing sinusoidal function' tasks, the sketches indicating the sinusoidal graphs were given and the student was asked to identify the sinusoidal functions represented in the given graphs (see Figure 1). For the 'Assigning coordinate' tasks, a wavy displace (see Figure 2) along with the sinusoidal functions were given and Emma was required to assign coordinates on the wavy curve such that it described the given functions. Type A tasks comprised of Task 1: $f(x) = sin(2x)$, Task 2: $f(x) = sin(\frac{2}{3}x)$ and Task 3: $f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$. Type B tasks included Task 4: $f(x) = sin(4x)$ and Task: 5 $f(x) = cos(3x - \frac{\pi}{4})$.

Theoretical Framework

The collected data in this study were analyzed and interpreted using the *theory of shifts of attention* (Mason, 2005*).* Mason's theory provides opportunity to study the critical role of attention and awareness in learning and understanding mathematics and in particular the concept of the transformation of sinusoidal functions. Mason (2005) distinguishes five different structures of attention: 1) Holding wholes; 2) Discerning details, 3) Recognizing relationships; 4) Perceiving properties; and 5) Reasoning on the basis of agreed properties. Mason's framework of shifts of attention is appropriate for analyzing the collected data in my research. Applying this framework supports me in gaining insights not only into '*what*' Emma attended to when completing mathematics tasks related to the transformation of sinusoidal functions, but also '*how*' she shifted her attention in identifying the period of sinusoids. Mason's terms for different structures of attention also provide a language for analyzing students' work. For example, when a student considers a particular graph and recognizes its shape as representing a sinusoidal function, s/he is *holding holes*. A student who looks for particular details from the given sinusoidal functions or the given sinusoidal

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

curve (e.g., she is seeking for the point the graph intersects the y-axis), she is, in fact, *discerning details*. The student is *recognizing relationship* when she able to find a connection between the graphical representation of the sinusoidal functions and their symbolic representations. When a student determines the particular parameters that determine the given sinusoidal curve by considering its periods, she is *reasoning based on perceived properties*. To investigate how the participant realized the transformation of sinusoidal functions, and in particular, how she identified period from the given graphs/functions, I reviewed the student's answers and the transcripts several times.

Please note that in all the five interview tasks the participant was required to connect the period of the given sinusoidal function or the given sine curve to a coefficient of x in the standard formula for sinusoidal functions (considering the sinusoidal function in the standard form: $f(x) = Asin(Bx \pm C) \pm D$)). For brevity, we refer to this connection as 'recognizing the period' (see Figure 3).

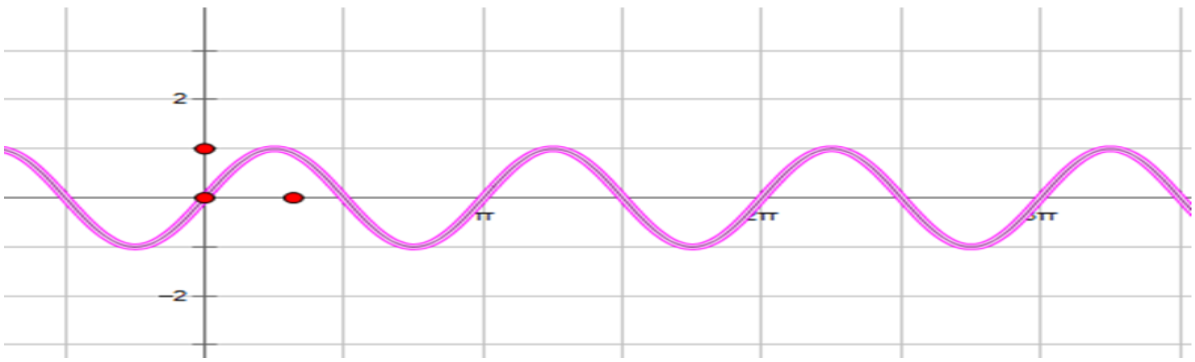

Figure 1. Graph presented in Task 1.

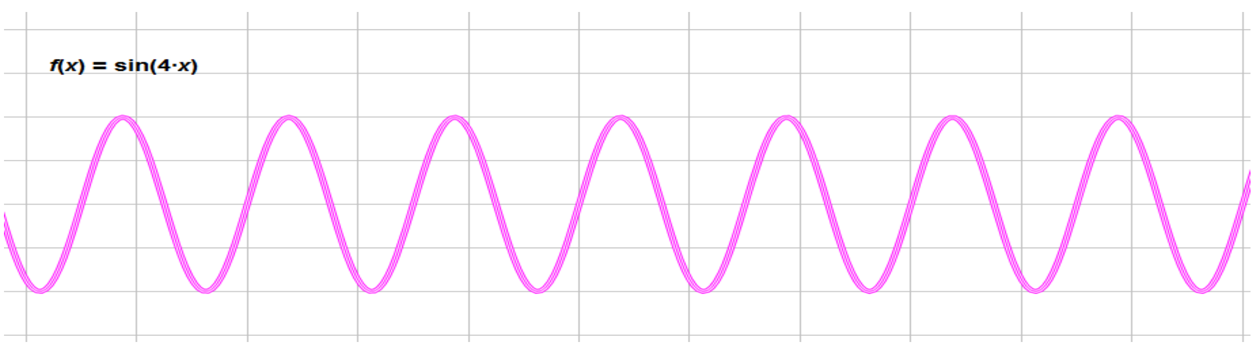

Figure 2. Graph presented in Task 4.

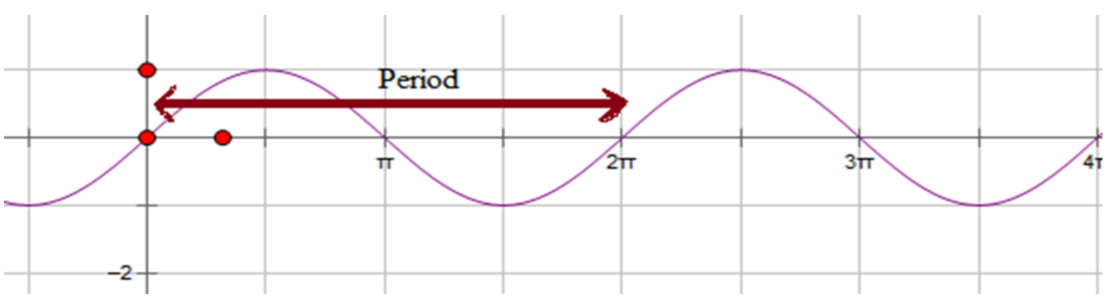

Figure 3. Recognizing the period.

Recognizing period (coefficient B of x)

At the beginning of the interview, I showed Emma Task 1 in which the graph of the function $f(x) = sin(2x)$ was given (see Figure4) and she was asked to identify the sinusoidal function represented by the graph. In order to complete Task 1, Emma first focused her attention on the given graph and waited for visual feedback from the graph (her attention was on *holding wholes* according to Mason's classification). Emma stated:

> It is $f(x) = sin(\frac{1}{2}x)$. It is a sine graph because it starts at 0 and it should be $sin(\frac{1}{2}x)$. The sine graph start at 0 and then π and 2π, but in this one is 0, π, $\frac{2\pi}{3}$. This is half of sine graph, because the period here is π, while it is 2π in the original sine curve.

The above statement indicates that the participant recognized incorrectly the function for the given graph, determining it to be $f(x) = sin(\frac{1}{2}x)$. Analyzing the situation using Mason's (2008) framework it can be concluded that Emma *reasoned on the perceived properties* of the sinusoidal functions and from there she *determined (*incorrectly*) relationships* between the visual representation and the symbolic representation. Emma recalled the fact that the period of a canonical sine function is 2π, whereas the period of the curve given in Task 1 was π . She thus concluded that the given curve represented the function $f(x) = sin\left(\frac{1}{2}x\right)$. Emma then connected the period of the sine curve, which was π radians, with the coefficient of x in her suggested sinusoidal function. Her statement illustrates that Emma, in fact, divided the argument x by 2 because the period of the canonical function (2π) was divided by 2 in the given graph (the period was π).

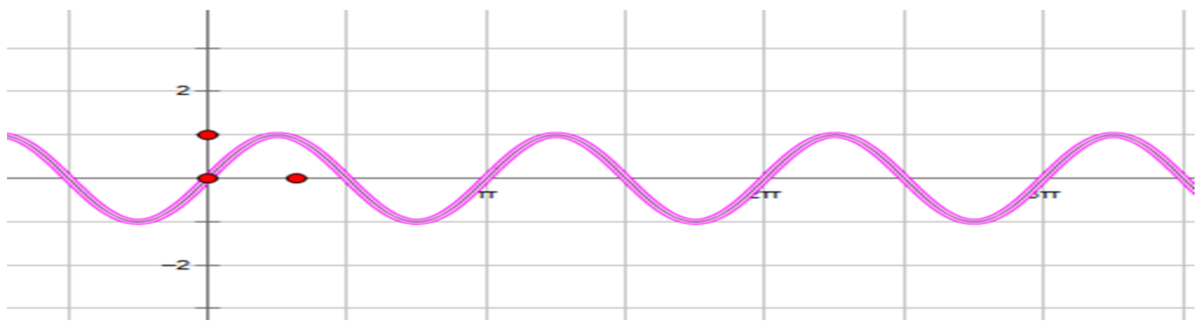

Figure 4: Graph of function $\boldsymbol{f(x) = sin(2x)}$.

Detecting Emma's mistake in recognizing the proper function for the given graph in Task 1, I showed her the graph of $f(x) = sin(\frac{1}{2}x)$. Observing the graph of the function $f(x) = sin(\frac{1}{2}x)$ made the participant realize that the graph of the suggested function did not correspond to the given curve. At this time Emma stared at both graphs #1 and #2 (see Figure 5) for a while and she *held the graphs (#1 and #2) as wholes*. She then began to *describe in detail* the given graph (#2 in Figure 5) in respect to the graph of $f(x) = sin(\frac{1}{2}x)$. Emma stated:

> ….so, if $f(x) = sin(\frac{1}{2}x)$ is like this, so it is going to finish at 4π. So this is going to be the whole graph. So it should not be $\frac{1}{2}x$, it should be 2x. Because when we have $\frac{1}{2}x$ we can see that it ends at 4π. But if I put here 2x, I compressed it and I can…have this curve finishes at π π...The period of sine graph is 2π but this one is compressed, so it is $f(x) = sin(2x)$, but $\frac{1}{2}x$ is expansion in fact.

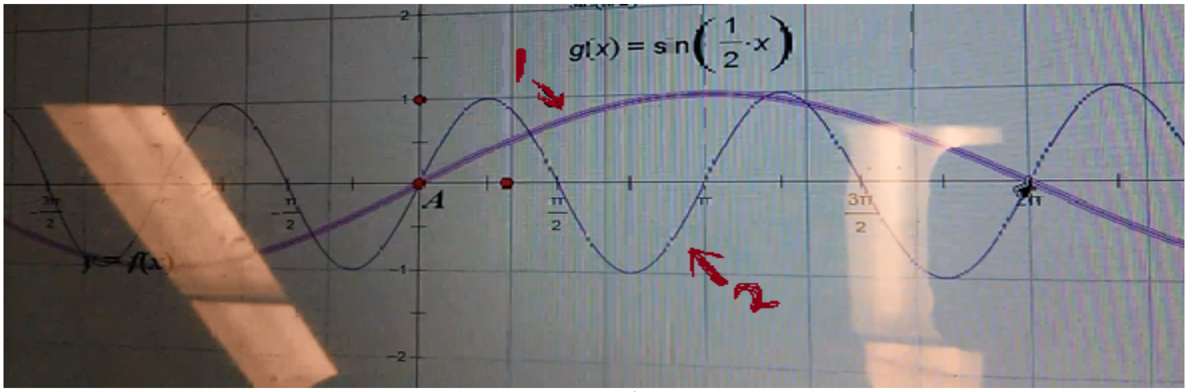

Figure 5: Graphs of $\boldsymbol{f(x) = sin(\frac{1}{2}x)}$ and $\boldsymbol{f(x) = sin(2x)}$.

As it is indicated from the above statement, Emma compared the end point (or the length of a full cycle) of the curve #2 with that of curve #1, considering the origin (0, 0) as a beginning of a cycle (*"....$\frac{1}{2}x$ we can see that it ends at 4*π. *But…I can…have this curve finishes at π...*"). In other words, by linking the end points of the full cycles (in both curves and comparing them with the graph of the canonical function), Emma was able to *find relationship* between the visual representation and the symbolic representations. She chose the number 2 (which was the reciprocal of the coefficient of $\frac{1}{2}\pi$) as a coefficient for x in the sinusoidal function. As such, she eventually recognized the correct period and thus the proper function for the given curve.

Emma's proper realization in Task 1 directed her to complete successfully similar tasks having a whole number for the coefficient of x. As an example, when approaching Task 4 in which the function was $f(x) = sin(4x)$ and a wavy curve was given, Emma was able to assign correctly the coordinates in the given wavy displace such that it represents the graph of $f(x) = sin(4x)$. After gazing at the given function in Task 4, she expressed:

> …I know that 2π is here [see Figure 6] because 1, 2, 3, and 4 periods is between 0 and 2π and here are 1 and -1…

The above excerpt shows that Emma perceived *properties of sinusoidal functions* ("*...period is between 0, 2π and here are 1 and -1*"). The feedback she received from Task 1 (the fact that there is a direct relationship between the coefficient of x in the sinusoidal functions and the number of repeated full sine cycles between 0 and 2π) allowed Emma to assign coordinates properly in Task 4. In other words, Emma was able to realize period from the given function and therefore assign axes successfully on the sinusoidal curve. Considering Emma's success in Task 1 and Task 4, one might conclude that she was able to recognize period and also sinusoidal functions, from their graphs, and vice-versa, successfully. However, Emma performed differently on the other interview tasks.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

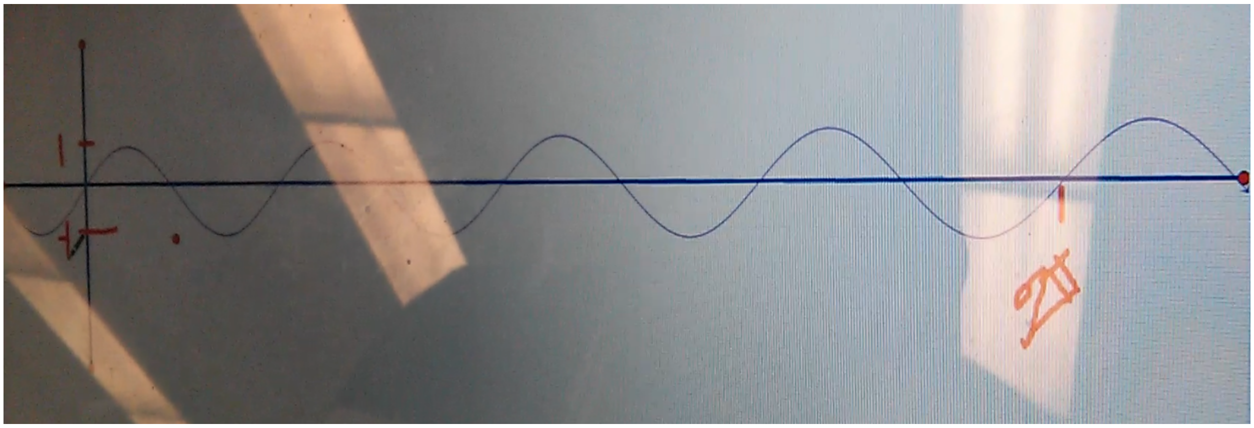

Figure 5. Emma adjusts coordinates for Task 4.

As an example, when completing Task 2, in which the graph of the function $f(x) = sin(\frac{x}{3})$ was given, after *holding the graph as whole* for a long pause, Emma did *discern some details* from the x-axis. She then stated:

> …It is sine of x over something because if it is sine of x it would end here [at 2π]…ok, it is $f(x) = sin(\frac{x}{3})$ because there are one, two and three spaces here between 0 and this point and again one, two, three here… (see Figure 6).

As it appears from the above statement, Emma counted the number of 'blocks' between 0 (the point A in Figure 6) and the point in which the curve intersected the x-axis (point B) and again from point B to another point in which the graph intersected the x-axis (point C). Since the distance between the points A and B, and B and C was 3 blocks, Emma put the fraction $\frac{1}{3}$ for the coefficient of x in the suggested sinusoidal function. It appears that she was eager to find an opposite relationship between 3π and the coefficient of x which was $\frac{1}{3}$. This evidence illustrates that Emma was unable to *recognize appropriately the relationship* between graphical representations of the sinusoidal function and its symbolic representations.

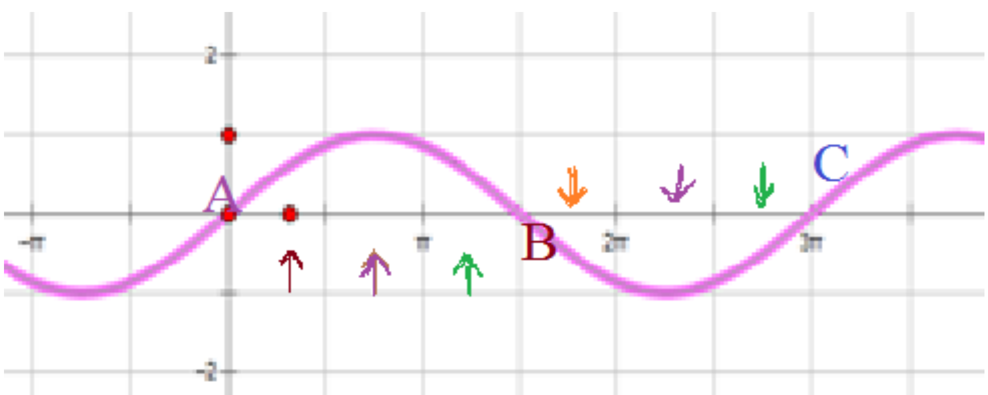

Figure 6. Emma counting the blocks between the points.

Emma's unsuccessful attempt in recognizing period and its relation with the coefficient of x in the sinusoidal function in Task 3 was typical of further errors in the other tasks having fractions for the coefficient of x. In other words, in Tasks 3 and 5, as in Task 2, Emma was unable to recognize

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

period successfully. As it was mentioned previously, it seems that the fractional coefficient was problematic, because Emma often attempted to reverse the point in which a full curve was finished (which was 3π in the Task 3) in order to find a coefficient for x in the sinusoidal function. Although applying this method directed Emma to determine proper functions when the coefficient of x was a whole number, it did not work for the other tasks. Emma, in fact, should find the relation between the period of a canonical function which is 2π and the point 3π in the given graph ($3\pi = \frac{2\pi}{B}$, so $B = \frac{2}{3}$) in order to identify the coefficient of x in the sinusoidal function.

Discussion

The findings of this study show that the student recognizes period and transformations in different manners when the coefficients of x in the sinusoidal function are whole numbers and when they are fractions. The data from this research demonstrates that Emma is capable in matching the algebraic representations with the graphical representations when the coefficient of x was a whole number. These results are in contrast with the findings of Leinhardt, Zaslavsky and Stein (1993), Yerushalmy and Schwartz (1993), and Knuth's (2000) studies in which a group of undergraduate students were unable to use graphical representations to complete mathematics problems in the symbolic form. The contribution of this research is in connecting together the participant's understanding of transformations, graphs and periodicity, whereas the previous research studies focused distinctly on the concepts of transformations (e.g., Yerushalmy and Schwartz, 1993), graphs (Brow, 2005) and periodicity (van Dormolen and Zaslavsky, 2003).

The findings, however, illustrate that Emma was unable to recognize period correctly, when the factor of x was not a whole number in the sinusoidal functions. That is, she was unable to connect the graphs with the sinusoidal functions when the factor of x was a fraction. As such, further research studies are required to investigate how undergraduate students interconnect the three concepts of transformations, graphs and periodicity when the coefficient of x is a fraction.

References

First Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS Theory: A framework for research and curriculum development in mathematics education*. New York, Heidelberg Dordrecht London: Springer.

Asiala, M., Brown, A., De Vries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and development in undergraduate mathematics education. *Research in Collegiate Mathematics Education*, *2*, 1-32.

Baker, B., Hemenway, C., & Trigueros, M. (2000). On transformations of basic functions. In H. Chick, K. Stacey, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference on the Future of the Teaching and Learning of Algebra* (1), 41–47. University of Melbourne.

Brown, S. A. (2005). *The trigonometric connection: Students' understanding of sine and cosine*. (Unpublished Ph.D. Dissertation), Illinois State University: USA.

Demir, O, (2011). *Students' concept development and understanding of sine and cosine functions.* (Unpublished doctoral dissertation), University of Van Amsterdam.

Lage, A. & Gaisman, M. (2008). An analysis of students' ideas about transformation of functions. *Advanced Mathematical Thinking*, (2), 23.

Kang, O. (2003). *A new way to teach trigonometric functions.* Sungkyunkwan University, Republic of Korea. Retrieved from http://www.icme-organisers.dk/tsg09/OkKiKang.pdf.

Moore. K. C. (2010). *The role of quantitative reasoning in precalculus students learning central concepts of trigonometry*. (Unpublished Doctoral Dissertation), Arizona State University.

Ng, K., and Hu, C. (2006). *Use web-based simulation to learn trigonometric curves*. Retrieved from http://www.cimt.plymouth.ac.uk/journal/chunhu.pdf.

Orhun, N. (2001). Students' mistakes and misconceptions on teaching of trigonometry. In A. Rogerson (Ed.), *The 21st Century Project Proceedings of the International Conference New Ideas in Mathematics Education International Conference on New Ideas in Mathematics Education* (pp. 208-211). Palm Cove: Australia.

Ross, J., Bruce, C., and Sibbald, T. (2011). Sequencing computer-assisted learning of transformations of trigonometric functions. *Teaching Mathematics Applications*, *30*(3), 120-137.

Weber, K. (2005), Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, *17*(3), 91-112.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.