THE ROLE OF DIAGRAMMATIC REASONING IN THE PROVING PROCESS

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The paper focuses on student-teachers' geometric diagrams to mediate the emergence of different proofs for a geometric proposition. For Peirce, a diagram is an icon that explicitly and implicitly represents the deep structural relations among the parts of the object that it stands for. Geometric diagrams can be seen as epistemological tools to understand explicit and hidden geometric relations. The systematic observation of and experimentation with geometric diagrams triggers abductive, inductive, and deductive reasoning which allows for the understanding of the conditions given for a geometric construction and its necessary logical consequences. We adopt Stjernfelt's model of diagrammatic reasoning to analyze two proofs for a geometric task posed to student-teachers who participated in a four-month classroom teaching experiment.

Keywords: Geometry and Geometrical and Spatial Thinking, Reasoning and Proof

Introduction

Peirce conceptualizes *sign* as a holistic triadic entity (*object*, *sign-vehicle*, *interpretant*). The word *sign* is sometimes used to refer to the *object* itself and, other times, to the mode of representation of that o*bject*. Peirce (1906) makes reference, through the paper, to the sigh being a general as to its object and as to its matter. The reader, then, is left with the task of interpreting either meaning from the context in which the word *sign* is used. His addition of the *interpretant*, as the third component of the sign, is one of his many significant contributions to semiotics. This component takes into account the effect of the sign-vehicle in the mind of the Person who interprets, uses, or produces it.

Peirce also classifies sign-vehicles as icons, indexes, and symbols according to their relation with the *object* they stand for. Fisch (1986) argues that this triad is not an autonomous species of sign-vehicles as if it were dogs, cats, and mice. Rather, it is a nested triad in which more complex sign-vehicles contain and involve specimens of simpler ones. Symbols typically involve indices which, in turn, involve icons. In other words, icons are incomplete indices which are, in turn, incomplete symbols.

The icon is a sign-vehicle that bears some sort of resemblance or similarity to its *object*. Peirce subdivides the icons into three types: images, diagrams, and metaphors. He argues, icon-diagrams have structural similarities with the structure of their *Objects*. This enables the observation, experimentation, and the emergence of inferential reasoning. He calls this amalgamated thinking *diagrammatic reasoning*. The index, instead, has a cause-effect connection to its *object*, and it directs the attention to its *object* by blind compulsion that hinges on association by contiguity (CP 1.558, 1867). The symbol, instead, hinges on intellectual operations, cultural conventions, or habit (CP 3.419, 1892).

> By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, performs experiments upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses it in general form. (CP 2.96, 1902)

Diagrams as Tools for Inferential Thinking

Peirce defines icons far beyond their merely perceptual aspects: "A great distinguishing property of the icon is that by the direct *observation* of it other properties concerning its *object* can be

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discovered than those which suffice to determine its construction" (Peirce 1895, Quoted in Stjernfelt, 2007, italics added). He clearly establishes diagrams as icons and as the only sign-vehicles from which more can be learned about the *object* beyond the grammar and syntax of their construction. While physical diagrams remain in the field of perception, new relations among their parts can possibly emerge by means of thought-experimentation and imagination. A diagram, then, can be characterized in one's mind in a variety of ways, "…as a token, as a general sign, as a definite form of relation, as a sign of an order in plurality, i.e., of an ordered plurality or multitude" (Robin 1967, Catalogue number 293, p. 31). The diagram, being an icon, has some kind of similarity with its *Object* in the sense that it displays the interrelations between the parts of the *object* in a skeleton-like sketch (Stjernfelt, 2007).

Peirce also argues that "the iconic diagram and its Initial Symbolic Interpretant constitute what… Kant calls *schema*, which is on one side an object capable of being observed while on the other side is a General" (NEM, p. 316) and that more can be learned about its *object* by contemplation of the explicit and implicit relations hidden in the diagram. In fact, he considers that diagrams are *epistemological tools* for inferential thinking. According to him "all necessary reasoning is diagrammatic" (Robin 1967, Catalogue number 293, p. 31).

Being a student of Kant's, Peirce adopts and adapts Kant's concept of geometric construction: "such a construction is formed according to a precept furnished by the hypotheses; being formed, the construction is submitted to the scrutiny of *observation*, and new relations are discovered among its parts, not stated in the precept by which it was formed, and are found, by a little *experimentation*, to be such that they will always be present in such a construction" (CP 3.560, italics added). This operational definition entails that once an empirical diagram is constructed what follows is some kind of mental experimentation and inferential experimentation.

A classic example of inferential manipulation and experimentation is Euclidean geometry. "Euclid first announces, in general terms, the proposition he intends to prove, and then proceeds to draw a diagram, usually a figure, to exhibit the antecedent condition thereof" (NEM, p. 317). Nowadays, given the dragging mode of dynamic geometry environments, the manipulation of geometric figures is expedited and, with it, the possibility of intentional experimentation. The observation of variant and invariant relations among the elements of the figure facilitates the conjecturing of its properties as well as the process of proving or disproving them. Stjernfelt, a semiotician, who has dedicated articles and books to the analysis of Peirce's diagrammatology, extensively argues that his definition of icon is non-trivial. This definition, he argues, avoids the weakness of most definitions of similarity because of its connection to the notion of observation and inferential experimentation to discover additional pieces of information about the *object* it stands for.

> …all deductive reasoning, even simple syllogism, involves an element of *observation*; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a *complete analogy* with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts (CP 3.363, italics added).

Diagrammatic Reasoning Process

Stjernfelt captures the essence of the process of diagrammatic reasoning in Figure 1. This is to say, a process which is rooted in perceptual and mental activity to produce chains of inferences. This figure is especially useful for thinking about proving and problem-solving processes. This skeleton-like figure, which is an icon-diagram itself, synthesizes a manifold of relationships that amalgamate the construction of a diagram, the observation of structural relations among its parts, and the physical and mental manipulation to produce a chain of deductions so as to attain a conclusion.

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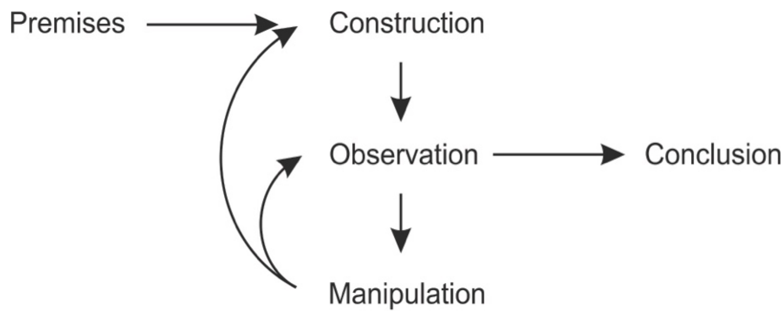

Figure 1. Diagrammatic reasoning process (Stjernfelt model, 2007).

In addition, Stjernfelt also describes this process in terms of the emergence of evolving interpretants generated during the transformation of diagrams. In this transformation, the implicit aspects of the *object* of the diagram are unveiled by means of the analogy between the relations among the characteristic properties of the object and the structural relations among the parts of the diagram. That is, the interpreting Person transforms icon-diagrams into sign-vehicles that have more and more symbolic aspects that hinge on mental operations and inferential reasoning. It is in this sense that "symbols grow," as Peirce says, because the meanings of their *objects* grow deeper and more general in the mind of the interpreting Person. A sequence of interpretants generated in this process is described by Stjernfelt (2007, p. 104) as follows:

a. Symbol (1)
b. Immediate iconic interpretant: Initial pre-diagrammatic icon-token that is rule-bound
c. Initial interpretant: (a+c) constituting the initial transformand diagram, the 'Schema' diagram-icon
d. Middle interpretant: the symbol-governed diagram equipped with possibilities of transformation (with two sources, **a** as well as **c)**
e. Eventual, rational interpretant: Transformate diagram
f. Symbol (2): Conclusion
g. A post-diagrammatical interpretant (different from **b**): This interpretant is an interpretant of **a**, but now, the diagrammatic reasoning is enriched by the total interpretant of the concept **a** [represented by Symbol (1)].

It is important to note that transformate diagrams are substantially contained in the transformand diagram with all its significant features. That is, diagrammatic reasoning is the process by which the interpreting Person intentionally endeavors both in the observation and manipulation of initial diagram-tokens (transformand diagrams) to mentally enrich and transform them (transformate diagrams) so that hidden relations among the parts of the *object* can be unveiled. These transformations facilitate the inference of the hidden structure of the *object*.

Methodology

A constructivist four-month classroom teaching-experiment on the teaching-learning of geometry was conducted with nine pre-service and in-service mathematics student-teachers who were taking a geometry methods course using the Geometer's Sketchpad (GSP). The main goal of this experiment was to improve student-teachers' ability to conjecture and to prove geometric propositions in plane Euclidean geometry using the GSP. An inquiry approach was used in which tasks were posed, drawings were constructed and manipulated by the students, conjectures were made, and proofs were generated. Student-teachers proved geometric statements in class and in homework assignments using this inquiry approach. They completed weekly homework assignments of at most seven tasks using the GSP. At the beginning of the semester, student-teachers were given a pre-test with twelve

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tasks to be solved using pencil and paper. At the end of the semester, they were given a post-test with thirteen tasks to be solved using the GSP. The purpose of these tests was to observe the influence, in the student-teachers' proving process, of the dynamic diagrams constructed, observed, and manipulated in the GSP environment.

Here we analyze, from the diagrammatic reasoning point of view, two proofs that pre-service student-teachers produced for task#1 in homework #9. This task could be proved in different ways: without using auxiliary lines and using auxiliary lines. The core of the geometric argument for each of the proofs was the conceptualization of congruent triangles embedded, implicitly or explicitly, in the diagram.

Data Analysis

In homework #9 student-teachers were given seven tasks. Task#1 was the following (see Figure 2):

Construct an isosceles triangle ∆ABC (AB ≅ AC). Extend the congruent sides BA and CA from the common vertex A and take AD = AE respectively. Let point M the midpoint of the base BC. **Prove that the triangle ∆MDE is an isosceles triangle.**

Figure 2. Task#1 given to student-teachers.

There are many different ways to prove this task but only one without the use of auxiliary lines. Only one student-teacher completed the proof this way. Four student-teachers constructed auxiliary lines EB and DC and completed the proof. Two student-teachers used the property that the median from the vertex-angle in an isosceles triangle is also perpendicular bisector. The other two student-teachers were unable to complete the proof because they constructed an incorrect drawing by extending the sides AB and AC from vertices B and C. The teacher wrote her proof and then presented it to the class.

First Proof

When the student explained her proof to the class, she showed on the computer screen the sequence of diagrams (Figures 3a, 3b, & 3c). This sequence conceals a sequence of interpretants, in the mind of the student-teacher (the interpreting Person), that allowed her a mental transformation of the same physical diagram (transformand diagram, i.e., Figure 3a) into transformate diagrams (Figures 3b & 3c) to conceptualize geometric relations. Finally, she presented the written proof of the given statement.

She constructed a robust isosceles ∆ABC using two radii of a circle centered on the vertex-angle A. In the extensions of sides AB and AC, from vertex-angle, she constructed congruent line segments AD and AE respectively. She continued with the construction of the midpoint M on base BC and of the ∆EMD (see Figure 3a). Then the student-teacher wrote down the given information $\overline{AB} \cong \overline{AC}$ and $\overline{AD} \cong \overline{AE}$ (see Figure 3b) as well as the proof that ∆BDM is congruent to ∆CEM by SAS. For the congruence of the triangles she explained that $\overline{MB} \cong \overline{MC}$ because point M is midpoint of base BC; $\overline{DB} \cong \overline{EC}$ because $\overline{DA} + \overline{AB} \cong \overline{EA} + \overline{AC}$; and $\angle ABC \cong \angle ACB$ as base-angles of isosceles triangle ∆ABC. The congruence of triangles ∆BDM and ∆CEM implies $\overline{DM} \cong \overline{EM}$ proving that triangle ∆DME is isosceles (see Figure 3c).

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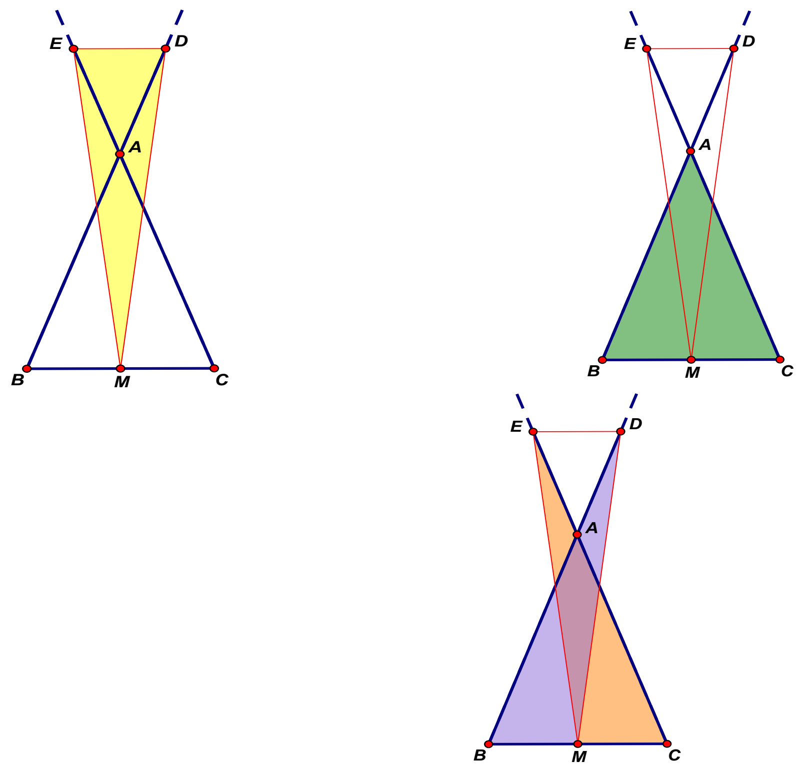

(a) **(b)**

(c)

Figure 3. (a) Initial construction; (b) Focus on isosceles triangles ΔABC and ΔADE; (c) Comparison of triangles ΔDMB and ΔEMC.

Following is also the description of the sequence of interpretants generated:

- <u>*Inmmediate iconic interpretant:*</u> Visual perception of two isosceles triangles ΔABC and ΔADE and the triangle ΔEMD (Figure 3a, i.e., transformand diagram).
- <u>*Initial Interpretant*</u>: A realization that adding congruent line segments ($\overline{AB} \cong \overline{AC}$ and $\overline{AD} \cong \overline{AE}$), by parts, would generate new congruent line segments ($\overline{DA} + \overline{AB} \cong \overline{EA} + \overline{AC}$ then $\overline{DB} \cong \overline{EC}$) (Figure 3b, i.e., transformate diagram).
- <u>*Middle Interpretant*</u>: ΔDMB and ΔEMC have two congruent sides ($\overline{DB} \cong \overline{EC}$ and $\overline{MB} \cong \overline{MC}$) and the respective angles between them are also congruent ($\angle ABC \cong \angle ACB$) being the base angles of isosceles ΔABC (Figure 3b, i.e., transformate diagram).
- <u>*Rational interpretant*</u>: Given that ΔABC is isosceles with $\overline{AB} \cong \overline{AC}$, the congruence of the base angles is implied ($\angle ABC \cong \angle ACB$). From the fact that point M is the midpoint of side BC, the congruence of BM and MC is also implied ($\overline{BM} \cong \overline{MC}$). Also using the congruence $\overline{DB} \cong \overline{EC}$ triangles ΔDMB and ΔEMC are congruent by SAS. (Figure 3b, i.e., transformate diagram).
- <u>*Eventual rational interpretant*</u>: Line segments $\overline{DM}$ and $\overline{EM}$ are the third sides of congruent triangles ΔDMB and ΔEMC; thus these line segments have to be congruent (Figure 3c).
- <u>*Post-diagrammatical interpretant*</u>: Consider triangles ΔEMC and ΔDMB. How are they related?
- $\overline{BM} \cong \overline{MC}$ (point M is the midpoint of $\overline{BC}$)

- $\angle ABC \cong \angle ACB$ (base-angles in the isosceles triangle ΔABC)
- $\overline{EC} \cong \overline{DB}$ ($\overline{EA} + \overline{AC} \cong \overline{DA} + \overline{AB}$ because $\overline{EA} \cong \overline{DA}$ and $\overline{AC} \cong \overline{AB}$ adding by parts)
- Therefore, $\Delta EMC \cong \Delta DMB$ by SAS and the corresponding sides $\overline{EM}$ and $\overline{DM}$ are congruent making triangle ΔEMD an isosceles triangle (Figure 3c, i.e., transformate diagram).

Her written proof certainly corresponds to the inferred intepretants from her diagrams (see Figure 4):

Given M is the midpoint of BC, ∴ CM = BM
Given isosceles △ABC, ∴ AC = AB and ∠C = ∠B
Given AE = AD, then AE + AC = AD + AB, ∴ EC = BD
Therefore, triangles △EMC and △DMB are congruent by SAS.
This implies that ME = MD making triangle △MED isosceles triangle.

Figure 4. First proof of Task#1 without auxiliary lines.

In the above proof, the student-teacher used direct relations between congruent corresponding sides and angles of isosceles triangles in the diagram to prove the congruence of triangles ΔEMC and ΔDMB. Then she implied the congruence of line segments $\overline{EM}$ and $\overline{DM}$. Thus, she proved that ΔEMD is isosceles.

Second Proof

The four student-teachers who used the auxiliary line segments $\overline{EB}$ and $\overline{DC}$ gave essentially the same proof (see Figure 5).

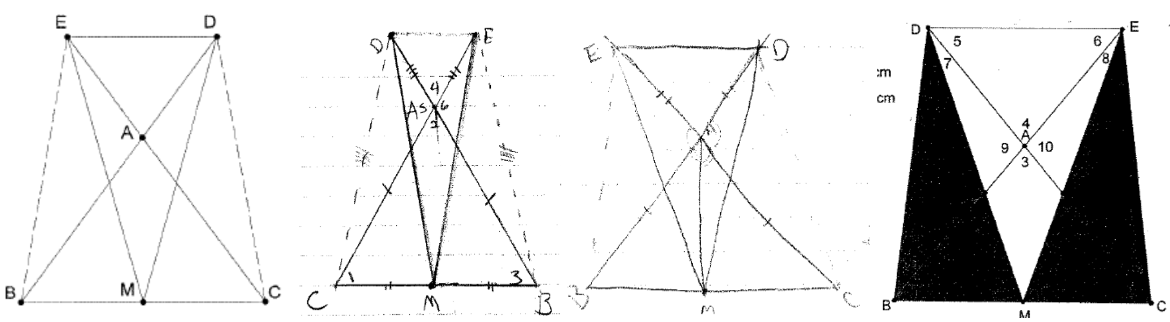

Figure 5. Drawings using the auxiliary line segments EB and DC.

Figure 5 presents the diagrams that four student-teachers gave using auxiliary lines EB and DC (or DB and EC). Below is the proof written by one of the four students, which corresponds to the first diagram on the left in Figure 5. The other student-teachers wrote similar proofs presenting the same argument (see Figure 6).

AB ≅ AC, AD ≅ AE, and BM ≅ CM through the construction. Also, ∠CAD ≅ ∠BAE (Task 1). Thus, by the Side-Angle-Side Theorem ΔABE ≅ ΔACD. Since ΔABE ≅ ΔACD and BE corresponds to CD, BE ≅ CD and ∠ABE ≅ ∠[redacted] = a. By Homework 8 ∠ABC ≅ ∠ACB = y. Thus ∠EBM = x+y = ∠DCM. Thus by the Side-Angle-Side Theorem, ΔEBM ≅ ΔDCM. Since ΔEBM ≅ ΔDCM, ME ≅ MD. Therefore ΔMDE is an isosceles triangle since two of its sides are congruent.

Figure 6. Second proof of Task#1 using auxiliary lines.

This student-teacher started with the relations $\overline{AB} \cong \overline{AC}$ and $\overline{AD} \cong \overline{AE}$ according to the construction of the isosceles triangles in Figure 5 and he also used the congruent vertical angles $\angle CAD$ and

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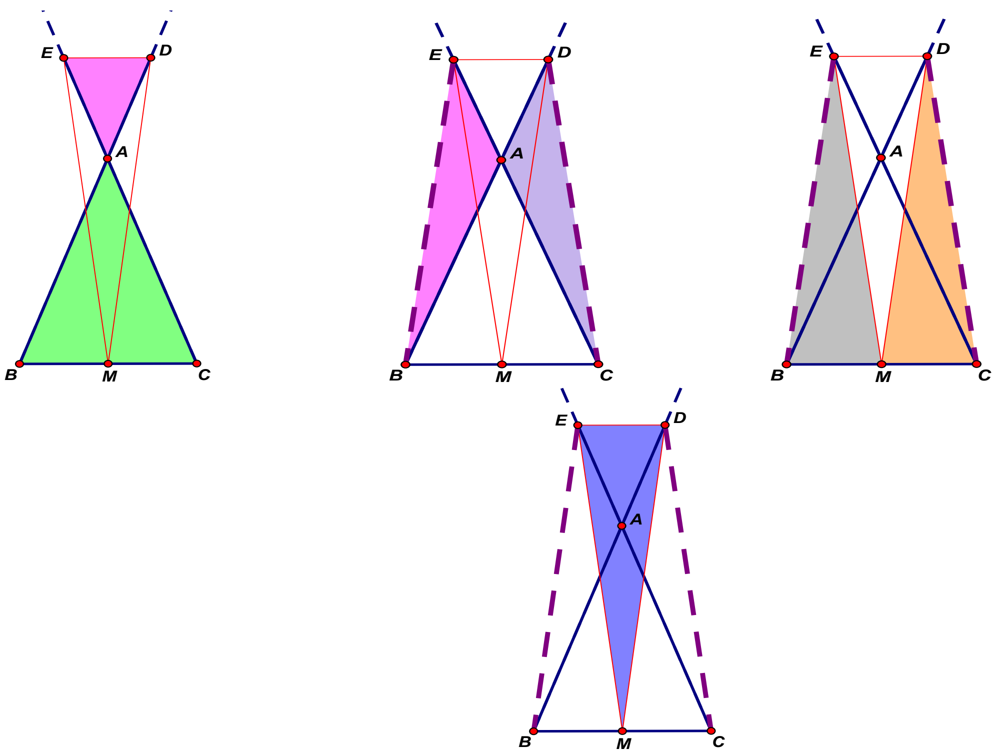
$\angle BAE$. He proved that triangles ΔABE and ΔACD are congruent by SAS. From this he implied that corresponding sides and angles are congruent ($\overline{BE} \cong \overline{CD}$ and $\angle ABE \cong \angle ACD$). Using the congruence of the base angles $\angle ABC \cong \angle ACB$ in the isosceles triangle ΔABC and the congruence $\angle ABE \cong \angle ACD$, he proved that $\angle EBM \cong \angle DCM$ as sums of congruent angles. Combining the relations $\overline{BM} \cong \overline{MC}$, $\overline{BE} \cong \overline{CD}$, and $\angle EBM \cong \angle DCM$, he proved the congruence of triangles ΔEBM and ΔDCM by SAS. An implication of the congruence of triangles ΔEBM and ΔDCM is that $\overline{EM} \cong \overline{DM}$. Thus, he concluded that ΔEMD is isosceles.

The above description can be unfolded into a sequence of diagrams that conceals a sequence of interpretants in the mind of the student-teacher (the interpreting Person). These interpretants allow for mental transformations of the same physical diagram (transformand diagram, i.e., Figure 7a) into transformate diagrams (Figures 7b, 7c, & 7d) to conceptualize other geometric relations and, finally, the proof of the given statement.



(a) (b)
(c) (d)

Figure 7. (a) Initial construction; (b) Comparison of triangles ΔABE and ΔACD created by auxiliary lines EB and DC; (c) Comparison of triangles ΔEBM and ΔDMC; (d) Conclusion triangle ΔEMD is isosceles.

- <u>*Immediate iconic interpretant:*</u> Visual perception of triangles (transformand diagram) constituted by two isosceles triangles (ΔABC and ΔADE) and also ΔEMD (Figure 7a, i.e., transformand diagram).
- <u>*Initial Interpretant*</u> A realization that drawing line segments $\overline{EB}$ and $\overline{DC}$ would generate two new sets of triangles: ΔEAB and ΔDAC & ΔEBM and ΔDCM (Figure 7b, i.e., transformate diagram).

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- *Middle Interpretant*: The segments $\overline{EM}$ and $\overline{DM}$ which are sides of ΔEBM and ΔDCM are also sides of ΔEMD. (Figure 7c, i.e., transformate diagram)
- *Rational interpretant*: A realization that $\overline{AB} \cong \overline{AC}$ are congruent sides of the given isosceles ΔABC. $\overline{AD} \cong \overline{AE}$ is a structural relation of the task. $\angle EAB \cong \angle DAC$ are vertical angles. Thus, ΔEAB and ΔDAC are congruent by SAS. Then both, the congruence of line segments $\overline{EB}$ and $\overline{DC}$ and the congruence of angles $\angle EBA$ and $\angle DCA$ are implied. (Figure 7b, i.e., transformate diagram)
- *Eventual rational interpretant*:
- How the line segments $\overline{EM}$ and $\overline{DM}$ could be compared? To which other triangles do they also belong? Triangles ΔEBM and ΔDCM are the triangles with the line segments $\overline{EM}$ and $\overline{DM}$ as sides (Figure 7c, i.e., transformate diagram).
- *Post-diagrammatical interpretant*:
- Compare triangles ΔEBM and ΔDCM
- $\overline{BM} \cong \overline{MC}$ (point M is the midpoint of $\overline{BC}$)
- $\overline{EB} \cong \overline{DC}$ (implication from the congruence of triangles ΔEAB and ΔDAC)
- $\angle EBM \cong \angle DCM$ (adding by parts congruent angles $\angle EBA \cong \angle DCA$ and $\angle ABM \cong \angle ACM$)
- Therefore, $\Delta EBM \cong \Delta DCM$ by SAS and the corresponding sides $\overline{EM}$ and $\overline{DM}$ are congruent making triangle ΔEMD an isosceles triangle (Figure 7d, i.e., transformate diagram).

In the above proof the student-teacher used auxiliary lines EB and DC. Then he used direct relations between congruent sides and angles to prove, first, the congruence of ΔEAB and ΔDAC and, then, the congruence of ΔEBM and ΔDCM. From the last congruence of triangles he implied the congruence of line segments $\overline{EM}$ and $\overline{DM}$. This proves that ΔEMD is an isosceles triangle.

Conclusion

The significance of *diagrammatic reasoning* in the teaching-learning of geometry during the proving process is analyzed in this paper. The Stjernfelt's model of *diagrammatic reasoning*, based on Peirce's own definition, was adopted. Essentially, diagrammatic reasoning consists of the systematic observation of a geometric diagram, the experimentation with the geometric diagram, and the inferential reasoning emerging from the observation of unveiled relations among the elements of the geometric diagram. Observation allows the visual perception of the explicit relations between the elements of the diagram. Experimentation with the diagram verifies these relations and facilitates the investigation of further relations. Finally, inferential reasoning emerges mediated by prior geometric knowledge and it makes possible the completion of the proving process.

Given the spatial and visual nature of Euclidean geometry, thinking and proving without diagrams seems to be an impossible task. Thus gaining awareness of diagrammatic reasoning as an epistemological tool appears to be useful for teachers to direct not only their own thinking but also the thinking of their students.

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