

EXPLORING STUDENTS' UNDERSTANDING OF THE LIMIT OF A SEQUENCE THROUGH DIGITAL AND PHYSICAL MODALITIES

Matthew Voigt

San Diego State University / University of California San Diego
MKVOIGT@gmail.com

The purpose of this study was to investigate how a set of physical and digital instructional activities can serve as an example space to help further develop a concept image that is aligned with the formal concept definition for the limit of a sequence. In addition, the unique affordances and constraints allowed by using either a physical or digital modality in understanding the convergence of a sequence was analyzed. Results suggest that both of these activities served to help students conceptualize the arbitrary nature of the error bound, and for some students it further illustrated the relationship between an arbitrarily small error bound, the limit value, and the index of the sequence. The physical activity constrained students to think of the sequence as a finite terminating set of numbers whereas the digital activity provided additional information that student used in subsequent problem solving.

Keywords: Technology, Post-Secondary Education, Instructional Activities and Practices

Students' misconceptions of limits and infinity are a well-documented and researched area in math education literature (e.g., Cory & Garofalo, 2011; Roh, 2008; Tall & Vinner, 1981). Students often have difficulty advancing from an intuitive dynamic conception of limit to the more formal limit conception, and often fail to understand the relationship between a given epsilon error range and the index of the sequence (Roh, 2010; Tall & Vinner, 1981). Students may resort to memorization tactics and often confuse the formal definition of the limit of a sequence with the limit of a function or other limit ideas.

Roh (2010) developed instructional materials that utilize visualization to support student understanding of the formal definition of the limit of a sequence based on the vast amount of misconception literature on limits and calls for the increased role of visualization in calculus (Dreyfus, 1990). Using an instructional activity referred to as the epsilon-strip task, Roh was able to categorize and support student understanding of the logical structure between epsilon and the index of a sequence. Roh has also shown how students' understanding of the definition of the limit of a sequence are influenced by their images of limits as asymptotes, cluster points, or true limit points using the same instructional activities (Roh, 2008).

This research project was driven by the conjecture that a technologically enhanced epsilon-strip instructional activity would make the instructional materials more accessible and promote deeper engagement by the user. Research shows that technology can be utilized to help students make sense of calculus concepts by using multiple representations (Tall, 1994) and that when technology uses dynamic versus static visualizations there is an overall benefit for student learning (Hoffler & Leutner, 2007). In a recent study with preservice teachers, dynamic sketches for sequence convergence were utilized to help strengthen their understandings of formal limit ideas as they integrated the visual representation with the symbolic definition (Cory & Garofalo, 2011). In this study, I seek to address how physical and digital epsilon-strip activities relate to students' understanding of limits of sequences. Furthermore, I investigate what constraints and affordances are provided in using either the physical or the digital epsilon-strip activity.

Theoretical Framework

Tall and Vinner (1981) examined student understanding of limits and developed the constructs of *concept image* and *concept definition* to describe the cognitive process of learning mathematics.

Concept image refers to the “total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes,” whereas the *formal concept definition* is the form of words which are accepted by the mathematical community to describe the concept (Tall & Vinner, 1981). Often times a student’s *evoked concept image*, which is the image generated given a certain context or prompt, may not be globally coherent and may deviate considerably from the formal concept definition. In the context of this study the formal concept definition for the limit of a sequence is referred to as the ϵ - N definition presented in Figure 1.

$\epsilon - N$ definition: A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L if, for any $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that for all $n > N$, $|a_n - L| < \epsilon$

Figure 1. Definition for the limit of a sequence referred to as the ϵ - N definition.

In addition to drawing on the concept image and concept definition framing, in this study I utilizes gesture theory from the field of embodied cognition as a means for illustrating student understanding through non-verbal communication. Preliminary research suggests that using dynamic visual environments can evoke metaphoric gestures to convey the temporal relationship in calculus concepts and are an essential element to effective mathematical communication (Ng, in press; Núñez, 2004). In this study I seek to answer the following research questions: In what ways do these instructional activities produce an evoked concepts image consistent with the concept definition for the limit of a sequence? Are there unique affordances or constraints in using either the digital versus the physical instructional activity?

Methods

The participants in this study were three undergraduate mathematics majors recruited from a Vector Calculus course at a 4-year public research university. At the time of the study, each student had completed a sequence of single-variable calculus and a course in analytic geometry and multivariable calculus. Additionally, each had previously been exposed to the definition of limit of a sequence and the concepts of convergence and divergence. The 45-60 minute semi-structured, individual interviews were video recorded and interview tasks were based on a modified sequence of instructional activities drawn from Roh (2010). Students were told they would explore a concept in calculus in a non-evaluative problem solving interview and were encouraged to think aloud as much as possible as they worked on each of the tasks.

In order to gain insight into each student’s initial evoked concept image for the limit of sequence they were asked about their prior experience with limits of sequences and how they thought about of the convergence of a sequence. To further explore student conceptions of convergence they were asked to represent the sequence $\left\{\frac{n^3}{2^n}\right\}_{n=1}^{\infty}$ numerically in a table and prompted to determine what would happen to the sequence as the index n gets larger. After assessing students reasoning and prior background regarding the convergence of a sequence they were presented with the following digital and physical epsilon-strip activities.

The digital epsilon-strip activities were created with the design heuristic principles of manipulation of content and guided discovery (Plass, Homer, & Hayward, 2009). The *manipulation of content* principle asserts that student learning is improved if the learner is able to manipulate the content of a dynamic visualization and the *guided discovery* principle suggests students learn better when guidance is used in discovery-based learning in multimedia contexts. Using these principles I created digital activities using Desmos[©] online graphing utility as shown in Figure 2. A link to the resources and graphs used in this study are available at the following web address (goo.gl/Bm31eD). Students were given a brief tutorial explaining the onscreen graphs and the epsilon-strip. In

particular, they were shown how to manipulate where the epsilon-strip was centered and how to adjust its width through the use of sliders, which provide dynamic changes of the values, or by manually entering a desired value. Based on the principle of guided discovery, students were given an activity sheet that instructed them where to center the epsilon-strip, and had them iterate through three decreasing values for the width of the epsilon-strip. For each given epsilon width, students filled out a table where they counted the number of terms inside the epsilon-strip and the number of terms outside the epsilon-strip, which is referred to as the *counting process*. Student were then asked to repeat the same procedure and counting process but with the epsilon-strip centered at a different value. There were a total of three graphs explored, each selected to illustrate a different type of sequence convergence or divergence: monotonically convergent $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, oscillating convergent $\{(-.75)^n\}_{n=1}^{\infty}$, and oscillating divergent $\{(-1)^n\}_{n=1}^{\infty}$.

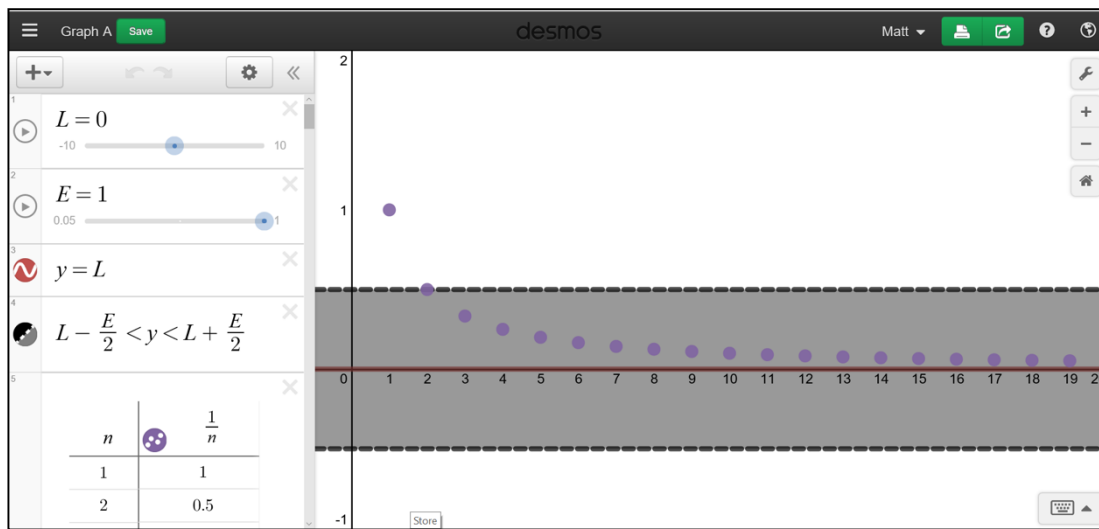


Figure 2. Digital epsilon-strip graph for monotonically convergent sequence.

The physical activity consisted of printed graphs of a sequence and strips of rectangular transparencies that represented the error bounds of a limit of a sequence, as shown in Figure 3. In accordance with the guidelines presented in Roh (2008) the strips had a constant width and each were marked through the center with a red line. The same procedure and counting process as described for the digital activity were used with the physical activity. There were a total of three graphs explored with this exact process: monotonically convergent $\left\{\frac{1}{n}, n \leq 10\right.$, oscillating convergent $\left.\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}\right.$, and oscillating divergent $\left.\left\{(-1)^n \left(1 + \frac{1}{n}\right)\right\}_{n=1}^{\infty}\right.$.

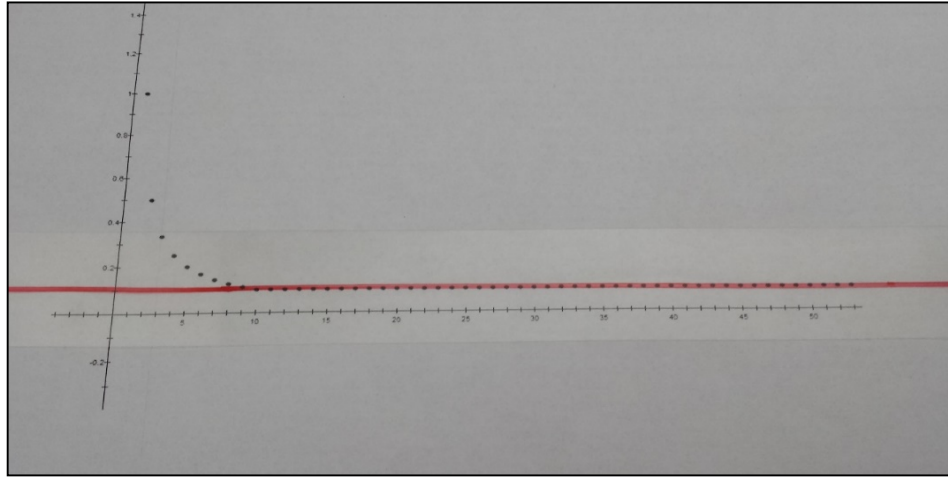


Figure 3. Physical epsilon-strip activity for monotonically convergent sequence.

After the digital and physical epsilon-strip activities, students were asked to evaluate the validity of two “student” generated definitions for the limit of a sequence, called ϵ -strip definition A and the ϵ -strip definition B:

- ϵ -strip definition A: L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any ϵ -strip as long as the ϵ -strip is centered a L .
- ϵ -strip definition B: L is a limit of a sequence when only finitely many points on the graph of the sequence are NOT covered by any ϵ -strip as long as the ϵ -strip is centered a L .

After evaluating these definitions students were given the formal ϵ -N definition and asked to explain each of the components of the definition and how they made sense of this definition. In the last part of the interview students were asked to explain how they saw the formal definition in relation to the ϵ -strip activities.

Results

Students’ evolving concept image

Pedro’s (all names are pseudonyms) initial concept image for the limit of a sequence was based on discrete representations, pattern recognition, and procedural computations. He recalled computing values for the limit of a sequence in prior mathematical classes, but was unable to provide a definition. When he was asked about the end behavior of the sequence $\left\{\frac{n^3}{2^n}\right\}$ he said it would depend on the relationship between the numerator and denominator. He stated that he would need to “plug in numbers and see the pattern” of values and then visualize those values to determine the limit. Pedro appeared to draw on a discrete representation of the sequence and pattern recognition to determine what the sequence was approaching as the index increased.

After the completion of the epsilon-strip activities Pedro was able to articulate several additional components related to his concept image for the limit of a sequence. He stated expressly the importance of where the epsilon-strip was centered, L , as crucial in determining if the sequence had a limit. He then linked the dependence on L with the idea that epsilon was allowed to vary and that for any epsilon there could only be finitely many terms outside of the strip. Although he didn’t formalize the directional relationship between any given epsilon and the existence of some index n , he did articulate a relationship between the arbitrary epsilon and the finite number of terms outside

the epsilon error range, and used this reasoning in rejecting ε -strip definition A. In describing the formal definition, he used the physical transparency showing how the sequence will be contained using gesture to show a motion like pattern in the strip, and then pointed to the digital epsilon-strip activity to show that it has to be bounded in that range. Although the activities elicited further components related to the limit of sequences, he was still unable to explain how definition B was valid, or articulate how each of the components in the formal definition related to the limit of a sequence.

Nick's initial concept image of the limit of a sequence was the most robust of all three students. Nick initially drew on the limit as an asymptote concept and described convergence as "it hitting a particular number as the index n approaches infinity." When computing the limit of the sequence $\left\{\frac{n^3}{2^n}\right\}$, Nick used a graphing calculator to plot the sequence as a continuous function and used the visualization to assert that the function was converging to zero. Nick's initial evoked concept image of the limit of sequence drew heavily from ideas of continuous functions and asymptotes versus the discrete representation conception evoked by Pedro.

After the completion of the epsilon-strip activities Nick's evoked concept image was similar to Pedro's yet contained further refinement regarding the relationship of the index n in the formal definition. Nick expressed that epsilon was "arbitrarily small" and you could, "pick any one" that you wanted, and that the limit value L was crucial in determining if the sequence converged to that particular value. Nick was the only student to explain the importance of the index n after picking an epsilon-strip, explaining that n is important because "we can always find an $n + 1$ such that it is inside our strip." Nick first rejected both definition A and definition B, using an oscillating divergent sequence as a counterexample to definition A. Nick further argued that definition B was an invalid definition because it was possible to have a finite terminating sequence and thus there would be finite number inside and a finite number outside the epsilon-strip. When invited to consider an infinite sequence, Nick utilized the physical graphs, and then stated that definition B would be valid.

Heng's initial evoked concept image for the limit of a sequence included categorization and computational techniques. When discussing the limit of a sequence he drew on ideas of divergence as infinity and convergence as "related to some sort of range" of values. When computing the limit of the particular sequence $\left\{\frac{n^3}{2^n}\right\}$, Heng said it would go to infinity because he used the ratio tests to compare the "values of a_{n+1} and a_n ." However, since he only compared the first two values of the sequence he failed to see that the sequence increases for the first few terms and then converges to zero.

During the physical epsilon-strip activity Heng viewed the oscillating convergent sequence as divergent since it alternated between positive and negative values. However, when working through the digital epsilon-strip activity he stated that the oscillating convergent sequence did converge since it was getting closer to the value of zero. Heng had one of the least developed concept images prior to the activity, and in addition expressed difficulty as an English language learner in engaging with the definition-provoking-activities that followed the digital and physical epsilon-strip activities. Heng initially asserted that definition A and B were correct, but after being asked to explain them in relation to the oscillating divergent sequence, he stated that A didn't fulfill the requirement and thus we "need both definitions for the limit" to exist. He did articulate that the epsilon-strip in the formal definition was arbitrary and that for "anyone that I pick" we can bound the sequence. He did not express the epsilon as arbitrarily small, nor was he able to express a relationship between epsilon and the index n . Heng, although acknowledging that they were different, wrote out the formal delta epsilon definition for continuous functions to try and make sense of the ε - N definition for convergence.

Affordances and Constraints

In addition to examining how the overall activities helped the development of student concept images related to the limit of a sequence, I analyzed the unique affordances and constraints provided by either using the digital or the physical modality for the epsilon-strip activity. During this study there were two prevalent constraints observed in using the physical modality related to the finite nature and imprecision of the epsilon-strips. Due to the nature of the static paper, both Pedro and Heng conceived of the graphs as a terminating sequence with only a finite number of terms and therefore had to be instructed to think of them as infinitely long. Although Nick did not evoke this misconception, he did use the example of a terminating sequence as a counter example to definition B, arguing that you could have finitely many inside and outside the strip. The imprecision of the epsilon-strips also lead each of the students to have questions about the ambiguity of terms lying on the epsilon-strip during the paper activity. This could have been a result of human cutting error regarding strip width uniformity or difficulties with the physical act of aligning and using strips. I speculate that the latter is more probable, since the strips were measured and cut using a guillotine style paper cutter and were observably uniform.

The digital modality offered several unique interactions that were not observed during the physical epsilon activity, including students using ancillary information provided by the digital modality and an increase in the use of gestures. Both Heng and Nick used the ancillary information that was located in the graphing utility on the left hand panel to reason and complete the activities. Heng, when working on the digital activities, first computed the epsilon error range given the width and center, and then used this range of numbers to compare against the table of sequence values in order to determine which terms were outside and inside the strip. Heng used this information to correctly identify the convergence for each of the digital activities, yet he failed to do so for the physical oscillating convergent sequence. Although this strategy was accurate, it is still unclear how this may have impacted his visualization of limits as it relates to the formal definition. Nick also drew on the additional information and argued that since the function was programmed in as $\{1/n\}$, he knew from prior experience that this sequence would converge to zero. It should be noted that the same ancillary information could have been included in the physical graphs, but they were an automatic result of using the digital technology.

Both Pedro and Nick increased their use of gestures when interacting with the digital modality and during the definition activity, as illustrated in Figure 4. I attended to the use of gestures in this study since the prior research from Roh's studies have failed to address the role of gesture in mathematical learning. Moreover, using the dynamic digital modality one might expect an increase in the use of gestures because the visual movement parallels the movement of the gesture. Pedro utilized his thumb and index finger in a collapsing motion when highlighting the arbitrarily small nature of epsilon while using the digital modality. Nick used a similar gesture when talking about the arbitrary nature of epsilon but used his thumb and all of his remaining fingers to show a collapsing dynamic motion. While neither Nick nor Pedro explicitly stated the concept that epsilon values tend toward zero, their gestures indicate an understanding of this very relationship. Nick also utilized his left forearm and right hand in an up and down motion when discussing the importance of where the epsilon-strip was centered in determining the limit while interacting with the physical modality, suggesting that the tangible movement of the strip was more salient given the physical context.

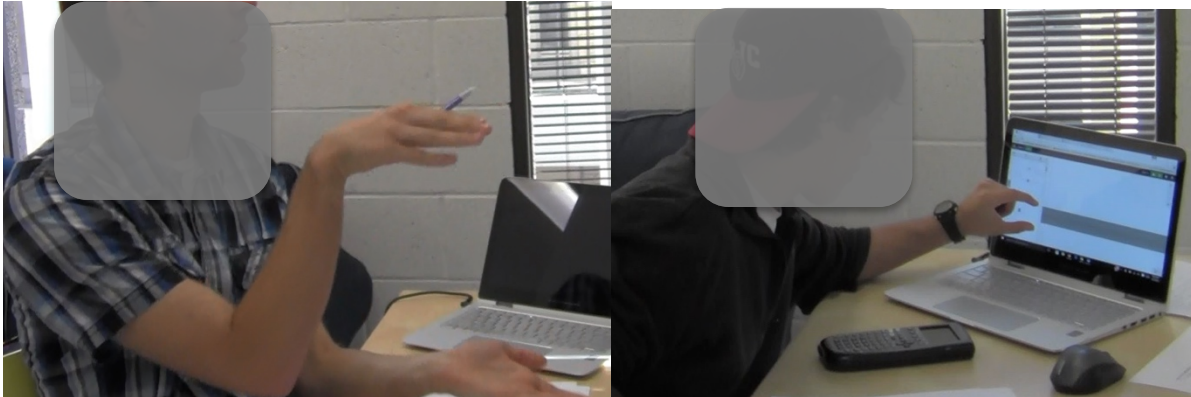


Figure 4. Nick (Left) and Pedro (Right) using gestures during the activities.

Conclusion

This study was driven by a desire to take existing innovative instructional materials and utilize technology in a controlled setting to ascertain how they may jointly help develop students' concept images for convergence, and furthermore provide insight into the affordances and constraints in using the two different modalities. Both the physical and digital activities helped to illicit a more detailed evoked concept images regarding the limit of a sequence from all three of the participants. All of the students after the activity expressed the limit in relation to the arbitrary nature of the epsilon-strip, and two of the students conceived of epsilon as getting arbitrarily small. Students still had issues explaining the formal ε - N definition of the limit of a sequence, with only one student articulating the importance of the index n as it related to the chosen epsilon error range. Although students didn't have a globally aligned concept image with the formal concept definition, each expressed how the nature of these activities helped them think and reason about the formal ε - N definition of the limit of sequence.

In comparing the two types of modalities, it appears that the physical activity seemed to constrain students into thinking of the sequences presented as finite and terminating. In addition there were issues of ambiguity related to when a given term was located inside the epsilon-strip either because they were not cut uniformly or because they were not centered level to the x-axis. The digital activity provided students with a dynamic representation which may have resulted in the increased presence of gestures in describing the definitions. In addition, the ancillary information was used in unique ways during the counting processes in unanticipated ways, such as the formal examination of the table of values. Future research studies may examine each of these activities with a larger range of students and determine if the gestures used by students is a unique affordance of the dynamic nature of the digital representation of the epsilon-strip activity and address the effect of the ancillary information (i.e. table of values or programmed function) on students understanding.

There is clearly more room for the examination of innovative and improved curriculum even though mathematics education literature abounds with research on student misconceptions of limits and infinity. In this study, a technologically enhanced epsilon-strip activity proved beneficial to promote student understanding of the limit of sequence. It provided a dynamic visualization of the epsilon-strip as an arbitrary small error range for a sequence that appeared to continue on ad infinitum. It is the hope that as these instructional materials and technology further develop, they will serve to broaden the example space that students draw upon in order to understand the formal mathematical definition of the limit of a sequence.

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