COHERENCY OF A TEACHER'S PROPORTIONAL REASONING KNOWLEDGE IN AND OUT OF THE CLASSROOM

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In this exploratory study we considered how one teacher's understanding of proportional reasoning related to his teaching. We used Epistemic Network Analysis to consider the teachers' knowledge organization and connections between knowledge resources as a way to make sense of his understanding. Then, we examined how his understanding was reflected in his teaching. From our analysis, we found key aspects of the teacher's proportional reasoning from two interviews related to his lesson. We concluded that this teacher's knowledge organization influenced his ability to teach lessons in coherent ways. This has implications for professional development.

Keywords: Teacher Knowledge, Rational Numbers, Mathematical Knowledge for Teaching

Purpose and Background

Research in mathematics education is beginning to show that the ways in which teachers understand mathematics matter (e.g., Baumert et al., 2010; Hill, Rowan, & Ball, 2005). How teachers understand their content is associated with what they are able to do when they teach (Ma, 1999; Silverman & Thompson, 2008; Thompson, Carlson & Silverman, 2007; Thompson, 2015; Wilson & Berne, 1999), and how that content knowledge is organized shapes their ability to teach coherently (Thompson et al., 2007). Drawing from cognitive psychology, we could frame the mathematics education findings as contributing to research on the development of expertise. Thus, in this paper, we cross the borders between cognitive psychology and mathematics education research. We also cross the borders between research on how teachers understand proportional reasoning and what that knowledge means to their practice.

We have chosen to focus on proportional reasoning because it is a salient domain of middle school mathematics and teachers are expected to support students in developing deep understanding in this domain. Proportional reasoning has become a prominent area, being treated as its own content domain in the Common Core State Standards for Mathematics (National Governors Association & Council of Chief State School Officers, 2010). Interestingly, despite the importance and its role as foundational knowledge for advanced topics in mathematics (Lamon, 2007; Lobato & Ellis, 2010), little is known about how teachers understand proportional reasoning (Lamon, 2007). The limited research suggests that, like students, teachers struggle with proportional reasoning (e.g., Akar, 2010; Harel & Behr, 1995; Orrill & Brown, 2012; Orrill & Kittleson, 2015; Post, Harel, Behr, & Lesh, 1988; Riley, 2010).

Teacher struggle may be related to the dominance of rote algorithms, such as cross multiplication, to solve proportion tasks instead of focusing on the multiplicative nature of proportional relationships (Berk, Taber, Gorowara & Poetzl, 2009; Lobato, Orrill, Druken, & Jacobson, 2011; Modestou & Gagatsis, 2010; Orrill & Burke, 2013). Studies have also suggested that teachers hold naïve conceptions about proportions (Canada, Gilbert, & Adolphson, 2008; Lobato et al., 2011). For example, Canada et al. (2008) found that only 28 pre-service teachers out of a sample of 75 were able to reasonably interpret a unit rate (e.g., amount per dollar) as useful for determining which package was a better buy when comparing two different size packages of ice cream. Teachers'

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proportional reasoning should include the understanding that a ratio represents a multiplicative comparison and not an additive comparison (Lamon, 2007; Lobato & Ellis, 2010; Sowder, Philipp, Armstrong, & Schappelle, 1998). This is a crucial understanding, as teachers need to be able to discern whether students are using additive or multiplicative reasoning (Sowder et al., 1998).

In this exploratory study, we examined how one teacher's understanding of proportional reasoning is related to his enactment of a lesson associated with proportional reasoning.

Framework

Knowledge in Pieces and Expertise

We rely on the knowledge in pieces theory (KiP; diSessa, 2006) to make sense of teachers' understandings. KiP asserts that our understandings are organized as fine-grained knowledge resources that can be drawn upon in a variety of combinations for a given situation. Learning occurs through perturbations that promote the development of new resources and the refinement of existing ones. Learning also includes developing connections between knowledge resources so they can be more readily drawn upon on in a variety of situations. KiP offers a unique lens for exploring the development of expertise, which is dependent, in part, on the extent of the *coherency of knowledge* (Orrill & Burke, 2013). We see coherency as meaning multiple knowledge resources connected in robust ways allowing for *in situ* access. Coherence, combined with a robust set of knowledge resources, allows teachers to deal with complex situations in more efficient ways. This is consistent with cognitive psychology research on expertise that shows that experts have both more knowledge than novices in their area of expertise and that their knowledge is organized differently than that of novices (e.g., Bédard & Chi, 1992). We also see our idea of coherency among knowledge resources as being consistent with Ma's (1999) concept of profound understandings of fundamental mathematics. We hypothesize that as a teacher's knowledge becomes more coherent (i.e., more knowledge resources are inter-connected), the teacher will be more flexible in supporting student learning of mathematics.

Epistemic Network

Epistemic Network Analysis (ENA; Shaffer et al., 2009) provides an analytical lens for identifying the connections between resources that a participant uses. We use ENA to focus on the connections participants make between knowledge resources, which are predefined using a coding scheme. Analysis is binary—thus each utterance either does or does not exhibit the presence of each pre-specified knowledge resource. ENA visually shows the relationships between the knowledge resources present (See Figure 1 for an example of a representation of an ENA equiload graph) in that it draws lines between those resources that co-occurred in a given utterance. We have interpreted these connections as being a way of determining connections between the resources. In this way, ENA provides a new alternative for measuring complex thinking and problem solving (Shaffer et al., 2009).

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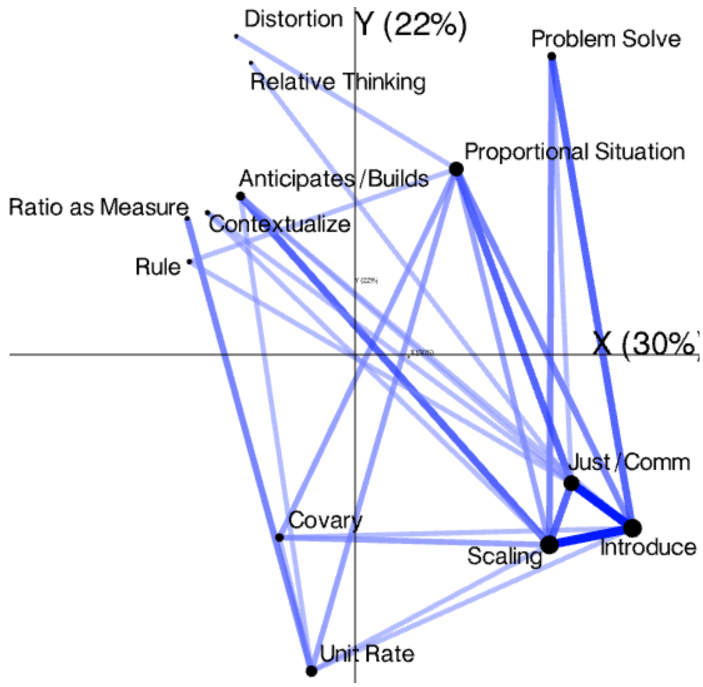

Figure 1. Matt's ENA equiload graph.

Methods and Data Sources

Data were collected as part of a larger project focused on teachers' proportional reasoning. Matt (pseudonym) was a certified 7th grade teacher with seven years of teaching experience. His classroom was in a K-8 school in an urban district in the United States.

Data were collected from two interviews and one classroom lesson. One interview relied on a paper-based protocol with 23 think-aloud prompts. Matt completed the protocol using a LiveScribe pen that recorded his voice as well as his written work. The second interview was a 90-minute videotaped clinical interview that included 18 items. The interview tasks were intended to elicit different aspects of reasoning about proportional relationships. The context of many items was in the work of teachers, asking participants to make sense of reasoning or work created by others. Both interviews were transcribed verbatim. The third data source was a video recording of Matt's 7th grade class during a single 90-minute lesson. We used two cameras: one focused on the primary speaker(s) and one directed on written work, when students were working in multiple groups a camera followed the teacher to document all teacher-student interactions.

We analyzed Matt's response to each interview task using a predefined code set for proportional reasoning knowledge resources (Table 1 shows all knowledge resources present in Matt's interview). We then created an ENA equiload graph for Matt's responses to all interview prompts (see Figure 1). In an ENA equiload graph, knowledge resources serve as vertices and the lines indicate those knowledge resources that co-occurred in the same response (see Table 1). The line thickness indicates the relative frequency of each co-occurrence across the interviews.

We relied on the same coding scheme to identify knowledge resources present in the classroom lesson. For the lesson, we coded knowledge resources present in each turn of Matt's talk. We then relied on qualitative analysis to compare Matt's understanding as portrayed in the ENA equiload graph to his understanding presented in the enacted lesson. (Note that we were unable to conduct an ENA-based analysis of the single class session due to mathematical limitations of ENA.)

Table 1: Knowledge resources present in Matt's interviews

Code	Description
Covariance	Recognizes that as one quantity varies in a rational number the other quantity must covary to maintain a constant relationship.
Ratio as Measure	Identifies an abstractable quantity created from the combination of the two quantities (e.g., flavor or speed) or discusses the effect of changing one attribute in terms of its effect on the ratio.
Unit Rate	Uses the relationship between the two quantities to develop sharing-like relationships such as amount-per-one or amount-per-*x*.
Scaling Up/Down	Uses multiplication to scale both quantities to get from one ratio in an equivalence class to another.
Relative Thinking	Demonstrates multiplicative reasoning about the change in a quantity relative to itself or another quantity. This includes re-norming.
Proportional Situation	Recognizes that a situation involves proportional reasoning.
Distortion	Describes "that things need to not get distorted" in similarity contexts.
Rules	Shares a verbal or written rule (e.g., blue = red + 2) stated in a way that conveys a generalizable relationship.
Anticipates or Builds from Others' Thinking	Talks about or builds from the mathematical thinking of others.
Contextualizing	Introduces a context for a relationship or anticipates impact of a context for students' reasoning.
Problem solving with Representation	Uses representation to support reasoning about the problem.
Justify or Communicate with Rep.	Justify or clarify a position already developed using the representation.
Introduce New Representation	Introduces a representation not implied or requested by task.

Results

In our analysis we found key aspects of Matt's proportional reasoning from interviews related to his lesson. Because of space limitations, our findings focus on Matt's knowledge organization and the ways it was reflected in his teaching.

Matt's Knowledge Organization

Matt's ENA equiload graph (Figure 1) showed strong connections between several pairs of knowledge resources. Strong connections indicate that Matt used those resources together in addressing particular tasks. We assert that co-occurrences of knowledge resources serve as indicators that the knowledge is linked in some way for the participant. Thus, we infer that connected knowledge resources indicate ideas that are conceptually tied together for the participant.

An example emphasizing strong connections is from the paper-based interview, in which Matt was asked to determine which was the better buy: a 16 oz box of Bites that costs \$3.36 or a 12 oz box of Bits that costs \$2.64. To solve this, he introduced a ratio table as a new representation (Figure 2) and used it to find a common multiple (link between *Introduce New Representation - Scaling Up/Down)*. Then, Matt communicated the idea that using ratio tables helps in "keeping it balanced and proportioned" (*Introduce New Representation – Justify or Communicate with Representation).* Further, Matt emphasized that a ratio table helps to show how the proportion is maintained

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(*Introduce New Representation - Proportional Situation*). In his solution, we see the 1s used by Matt to add 3.36 twice to get to 10.08 in his ratio table as opposed to him using multiplicative reasoning to solve the problem.

Matt was not limited to his own ratio table reasoning. He offered that students might try to solve Bits and Bites with long division and suggested that scaling is more efficient for them (*Scaling Up/Down -Anticipates or Builds from Others' Thinking*). He also explained that this task could be solved using unit rate (*Unit Rate*) when he said: "I could've also got down to unit rate—how much per ounce." Matt had access to this knowledge resource, but chose not to use it often in solving tasks. He explicitly talked about how unit rate can be difficult for students to use because of division. We assert that this special attention to *Unit Rate* is consistent with Matt's ENA equiload graph, which shows *Unit Rate* only weakly linked to *Scaling Up/Down* for Matt. This suggests Matt considers *Unit Rate* as an approach for only specific tasks.

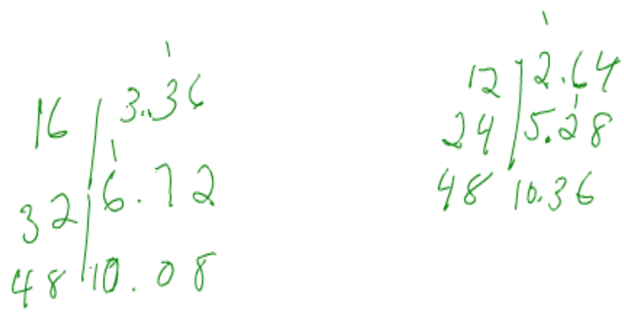

Figure 2. Matt's written work for the Bits and Bites problem.

Matt's Classroom

Matt's pattern of knowledge resource use was echoed in his teaching. Matt described the concept of proportion in his lesson the same way he had in his interview. He emphasized the importance of keeping the equivalence by *adding* the same amounts. This way of describing how to maintain a proportion suggested additive reasoning rather than multiplicative reasoning.

In Matt's lesson he posed a task similar to Bits and Bites. Students were asked to compare two deals on pencils. Matt set a clear objective "students will be able to determine the better deal using proportional reasoning and ratio table". This parallels the strong connection between *Introduce New Representation* and *Scaling Up/Down* consistent with his own problem-solving approach.

In the whole class discussion, Matt emphasized the use of ratio tables to solve the pencils problem by recognizing that the situation involved proportional reasoning. This was consistent with the approach we saw him take in Bits and Bites. However, Matt was able to make sense of students' different representations and solution strategies as well as how they communicated their solutions. For example, one student suggested another strategy using the common multiple of 120. Matt took a paper and pencil and tried it out himself by representing the pencil deals in ratio tables and scaling up both packs to 120 to solve the problem (this parallels his pattern of connecting *Introduce New Representation - Scaling Up/Down*). Matt also used students' work in his instruction, for example he used some students' work to discuss scaling using a ratio table versus long division (*Scaling Up/Down -Anticipates or Builds from Others' Thinking*).

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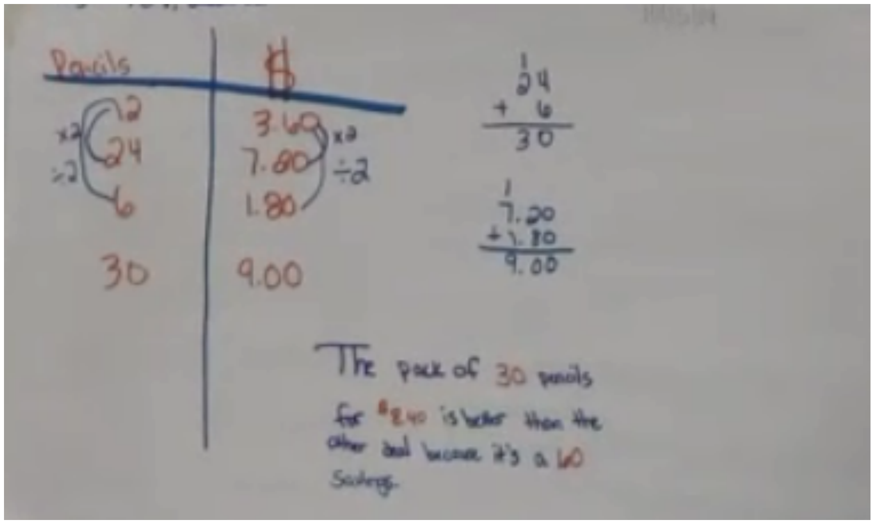

Figure 3. Student's work – using one ratio table to scale the 12-pack of pencils.

Consistent with his approach in the interviews, Matt did not rely on *Unit Rate* in his teaching. As students worked in small groups, they used ratio tables in different ways. Some built one ratio table (Figure 3) to scale only one pack of pencils and some used two ratio tables to scale both packs (Figure 4). And when students did attempt unit rate, Matt tried to steer them away. For instance, Matt approached one student who tried to use unit rate and asked if "there is a reason to get to one?" He then guided the student to think about getting to 60 instead. Further, when one group presented the solution using unit rate to the class (Figure 4), Matt said that finding unit rate is an efficient strategy "if the question asks you for the price per pencil". Matt seemed to prefer *Scaling Up/Down* using ratio tables in his teaching as well as in his problem solving. This was consistent with his interviews in which he mentioned that he worried about students relying on unit rate because it required strong division skills that his students often did not have.

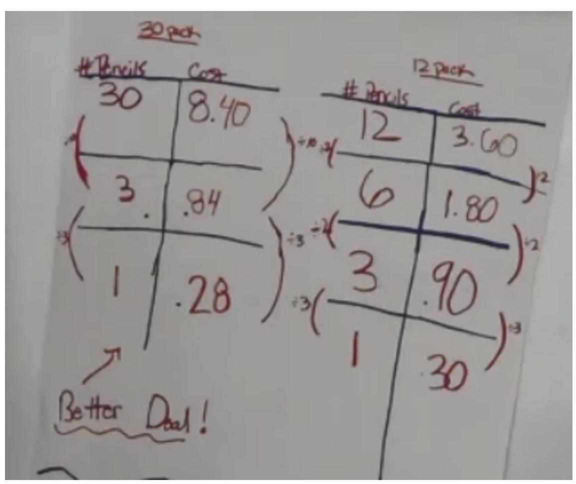

Figure 4. Student's work – using ratio tables to scale both packs of pencils to 1.

Conclusions

The knowledge resources and connections Matt used to solve problems about proportions were consistent with the ways in which Matt used the same resources and connections to guide his lesson. This aligns with the assertion that a teacher's relative level of coherence will shape their ability to teach lessons in coherent ways (Thompson, et al., 2007). We note that Matt's views of *Unit Rate*, his

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definition of proportion, and his reliance on ratio tables were factors that shaped his interaction with his students. Additionally, his use of additive language as opposed to multiplicative language when describing a proportion was used with his students and in his interviews. Matt understood unit rate and had taught his students unit rate; but he reinforced other approaches for problem solving in his classroom. One important take-away from this study is that despite showing evidence of having particular knowledge resources, Matt did not draw on them in his teaching. Thus, demonstrating particular measurable knowledge may not be a sufficient measure for teacher knowledge as they may have access to an array of knowledge that is not used in their classroom teaching. This raises questions about how best to measure or research teacher knowledge as it relates to opportunities for student learning.

Scholarly Significance

This study explores what it means for a teacher to have coherent knowledge. While many researchers assert that mathematics teachers need a deep understanding of the mathematics they teach (e.g. Ma, 1999; Thompson et al., 2007), little work has been done to understand what this means and what difference it makes to students' opportunities to learn. This study contributes to research on teacher understanding by stepping away from focusing on quantifying the amount of knowledge a teacher exhibits to instead focus on how the organization of that knowledge was used to drive the enactment of a single lesson. Studying how teachers understand and use the mathematics they teach has practical implications on the design of teacher preparation and professional development programs.

The teacher, in our study, demonstrated a variety of knowledge resources about proportional reasoning and strong connections between some of those resources. Based on our experience watching teachers implement proportional reasoning lessons, we hypothesize that not all teachers have the same kinds of connections between their own understanding and their teaching. We will be conducting additional analyses to determine whether the similarities between personal knowledge and enacted knowledge are maintained.

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