APPROPRIATENESS OF PROPORTIONAL REASONING: TEACHERS' KNOWLEDGE USED TO IDENTIFY PROPORTIONAL SITUATIONS

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In this study we explored to what extent middle school teachers were able to appropriately identify proportional situations when presented with various mathematical structures and if there were relationships between attributes of the teachers and their ability to identify proportional situations. Interestingly, there were no strong relationships aside from teachers' perception of their knowledge of mathematics and their ability to identify proportional situations. Teachers were also found to correctly identify proportional situations significantly more often than non-proportional situations. Nearly one third of the teachers misidentified non-proportional linear situations as proportional. Thirteen participants' responses to such a situation were analyzed qualitatively resulting in some common knowledge resources that they appeared to use when attempting to identify whether a situation was proportional or not.

Keywords: Teacher Knowledge, Rational Numbers, Mathematical Knowledge for Teaching

Purpose and Background

Proportional reasoning is an important content area that has gained prominence in middle school mathematics. One indication of this is that the Common Core State Standards for Mathematics (National Governors Association & Council of Chief State School Officers, 2010) have made "ratios and proportional reasoning" its own content domain for grades 6 and 7. Despite this recognition, there has been little focus on proportional reasoning in research in relation to its importance (Lamon, 2007), including research on teachers' knowledge of the domain. The research that is available on teacher knowledge of proportions indicates that, like students, teachers struggle with proportions (e.g., Akar, 2010; Harel & Behr, 1995; Orrill, Izsák, Cohen, Templin & Lobato 2010; Post, Harel, Behr, & Lesh, 1988; Riley, 2010).

One fundamental way of demonstrating proportional understanding is in the ability to identify when a situation warrants the use of proportional reasoning, which pertains to the mathematical structure of the problem rather than other identifiable aspects. Orrill et al. (2010) observed that middle school teachers had trouble identifying situations as appropriate or inappropriate for using proportional reasoning. For example, often when teachers were given a problem with three values and asked to find a missing fourth value, teachers tended to treat it as directly proportional even if the actual relationship was inversely proportional. Teachers also struggled to reason about proportions in a qualitative task that asked them to compare one pile of blocks to another pile similar to those tasks used by Harel, Behr, Post, and Lesh (1992).

These findings suggest that when teachers do not rely on a strong mathematical understanding of proportions to evaluate a situation, they draw on understandings that may be based on something other than mathematical structure when deciding whether a situation is proportional. In this paper, we explored the extent to which middle school teachers were able to identify proportional situations and whether there was a relationship between the teachers' backgrounds and their ability to identify proportional situations? To further explore this, we investigated whether there was a relationship between the underlying mathematical structure of the situation presented and the teachers' ability to identify proportional situations. We followed this with an analysis of the knowledge resources our participants invoked when identifying a non-proportional, linear situation. We see this work as border crossing because we are looking at the between proportions and other relationships. Specifically, we are investigating middle school teachers' abilities to recognize the border between situations that are and are not proportional. We conjecture that without such recognition teachers will struggle to use appropriate reasoning to make sense of a given situation.

Theoretical Framework

We work from the knowledge in pieces perspective (diSessa 1988, 2006), which asserts that individuals hold understandings of various grain sizes that are used as knowledge resources in a given situation. These resources are connected, over time, through learning opportunities that lead to the refinement of the resources and the development of rich connections. Having a series of robust connections allows a knowledge resource to be available in more situations. This is parallel to the research on expertise that has shown that experts have both more knowledge and a different organization of knowledge than novices in their domain (Bédard & Chi, 1992). It is also aligned with Ma's (1999) interpretation of teachers' need for profound understandings of fundamental mathematics. By having a robust set of knowledge resources that are coherently connected, we posit that teachers will be more able to access their myriad understandings to apply them to a wider range of mathematics and teaching situations than others whose knowledge resources are less coherently connected. We refer to this richly connected collection of knowledge resources as being *coherent* and assert that more coherent teachers will be better able to support student learning (e.g., Thompson, Carlson, & Silverman, 2007). This approach differs from much research on teacher knowledge in that we are not trying to identify deficiencies in teachers' understanding of mathematics, rather, we are trying to understand how teachers understand the mathematics they teach and how different knowledge resources are drawn upon for solving problems and teaching.

Methods

This study is part of a larger project investigating teachers' knowledge of proportional reasoning for teaching. In this section we will describe the participants of the study as well as our data collection and analysis procedures and tools.

The participants included a convenience sample of 32 in-service, grade 5-8 mathematics teachers, whose teaching experiences ranged from 1 to 26 years. The participants were from four states. They taught at a variety of schools (public, private, and charter). Twenty-four of the teachers identified as female and eight identified as male. Six of the teachers identified as a race other than white.

The data analyzed for this study were collected through a multiple-choice assessment and clinical task-based interview. We selected those items designed to assess teachers' ability to differentiate between situations that are or are not proportional. The items (n=20) were drawn from three existing assessments designed to measure teachers' proportional reasoning abilities. We also collected data on attributes of teachers' backgrounds including: the number of years teaching, number of mathematics and methods courses taken, and teacher's self-efficacy. To do this, we relied on seven, five-point Likert scale items taken from the Learning Mathematics for Teaching assessments (Learning Mathematics for Teaching, 2007).

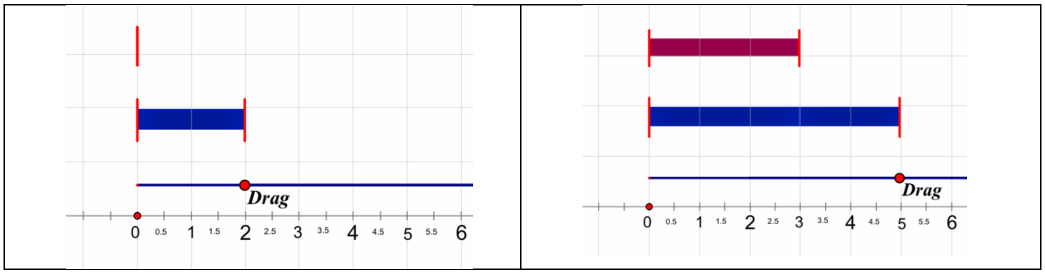

Figure 1. Screenshot of Thermometers task.

We carried out an exploratory data analysis of the proportional reasoning items, followed by a correlational analysis and a non-parametric comparison of group centers to investigate the first three research questions. The results of the analysis led us to analyze the Thermometers task from the clinical interview. Thermometers, a dynamic sketch, presented the participants with two thermometers, one red and one blue, whose lengths could be varied by dragging a point on a number line (as shown in Figure 1). Two scenarios were shown to participants (one at a time) and with each scenario participants were asked: (a) whether there was a relationship between the thermometers; (b) whether the relationship was proportional; (c) whether they could provide a rule and a story problem or real-world situation for that relationship; and (d) whether they see a scale factor involved in the situation. For this study we analyzed the participant's responses to the first scenario where the thermometers were designed to maintain a constant difference of two units in length of the lines as the point on the slider is dragged from left to right, shown in Figure 1. This situation represents a non-proportional linear relationship between the two thermometers.

Table 1: Codes of Knowledge Resources Used in Thermometers Task

Code	Description
Comparison of Quantities	States that ratio as a comparison of two quantities.
Covariance	Recognizes that as one quantity varies in rational number the other quantity must covary to maintain a constant relationship.
Unit Rate	Uses the relationship between the two quantities to develop sharing-like relationships such as amount-per-one or amount-per-x.
Equivalence	Describes proportion as a relationship of equality between ratios or fractions.
Between Measure Space	Asserts that the ratio between the quantities in a proportion stays constant.
Scaling Up/Down	Uses multiplication to scale both quantities to get from one ratio in an equivalence class to another.
Horizon knowledge	Demonstrates knowledge that extends into mathematics beyond proportions
Relative Thinking	Demonstrates multiplicative reasoning about the change in a quantity relative to itself or another quantity. This includes re-norming.
Proportional Situation	Recognizes that a situation involves proportional reasoning.
Rule	Shares a verbal or written rule (e.g., Red = Blue - 2) stated in a way that conveys a generalizable relationship.
One Unit at Time	Describes the relationship between the two quantities as increasing by one unit at a time

The qualitative analysis of the participant's responses was carried out by coding the participants' utterances using a coding scheme (Table 1). The scheme was developed using open coding (Corbin & Strauss, 2007) and refined across several interviews. It was specifically designed to consider knowledge resources related to proportional reasoning. We note that this specific task was non-proportional, so there were other resources that the participants drew on to engage with the thermometer scenario. Our coding relied on a binary approach in which each utterance was coded as a 1 or a 0 based on whether a particular knowledge resource was observed. Every interview was coded by at least two researchers and 100% agreement was reached on all coding.

Results

Question 1: Extent of Correct Identification of Proportional Situations

The 32 teachers in this sample were largely able to correctly identify proportional situations. For the 20 situations analyzed, the mean number correct was 15.22 (SD=2.97). The range of correct answers was 7-19.

Table 2: Kendall's tau b Correlation Coefficients

	# Items Correct					
	Kendall's tau b	Sig. (2-tailed)	Min	Max	Mean	SD
Years Teaching Math	.144	.272	0	26	9.20	7.202
Math Courses	.135	.352	1	4	3.13	.92
Methods Courses	-.012	.932	1	4	2.59	1.01
I enjoy teaching mathematics	.000	1.000	3	5	4.72	.581
Mathematics isn't my strongest subject to teach	-.271	.065	1	5	1.69	1.120
I consider myself a "master" mathematics teacher	.255	.070	1	5	3.03	1.177
Overall, I know the mathematics needed to teach this subject	.278	.063	3	5	4.34	.653
I have strong knowledge of ratio, proportional reasoning, and rate	.483**	.001	2	5	3.81	.738
I have strong knowledge of all areas of mathematics	.281*	.050	1	5	3.09	.963
My knowledge of ratio, proportional reasoning and rate is adequate to the task of teaching these subjects	.343*	.017	1	5	3.94	1.105

Note. Math and Methods courses measured on a scale of 1 to 4 with 1 corresponding to no classes, 2 to one or two classes, 3 to three to five classes, and 4 to six or more classes. *$p \leq .05$, **$p < .01$

Question 2: Relationship between Attributes of Teacher's Background and Ability to Identify Proportional Situations

In response to the second research question a correlational analysis was done using Kendall's tau B (shown in Table 2), which is a non-parametric measure of correlation appropriate for small samples (Field, 2013). Interestingly there was no significant correlation between the number of items the teachers answered correctly and the number of college mathematics or methods courses. There was also no significant correlation between the numbers of years teaching and the number of correct responses. However, there was a significant correlation between the number of items answered correctly and the teachers self-rating of their knowledge of mathematics; of ratio, proportional reasoning, and rate; and of perceived ability to teach these subjects. Further investigation in this area is needed as these results are not generalizable given the sample size.

Question 3: Relationship between Mathematical Structure and Appropriateness

In response to the third research question we looked to see if there was a relationship between the underlying mathematical structure of the situation and teachers' abilities to determine whether it was proportional. Using the non-parametric Mann-Whitney U-test we found that the median number of teachers who correctly identified the proportional situations was 31 for each of the seven items that was proportional. The median number of teachers who correctly identified the non-proportional situations was 23 for each of the 13 non-proportional items. Using the Mann-Whitney U-test, we

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found that this was a significant difference (U=3.00, z=3.34 p=.0002, r=.75). This means that the teachers were better at appropriately identifying proportional situations than they were at identifying non-proportional situations. Further analysis revealed a clear pattern in the relationship between the mathematical structure of the item and the number of teachers able to correctly identify it as proportional or not (Figure 2). There is a clear stratification of the number of appropriateness items that teachers responded to correctly based on the underlying mathematical structure of the situation presented. There is also a statistically significant correlation (Kendall's tau-b -.889, p<.0001) between the mathematical structure and the correctness of teachers' responses.

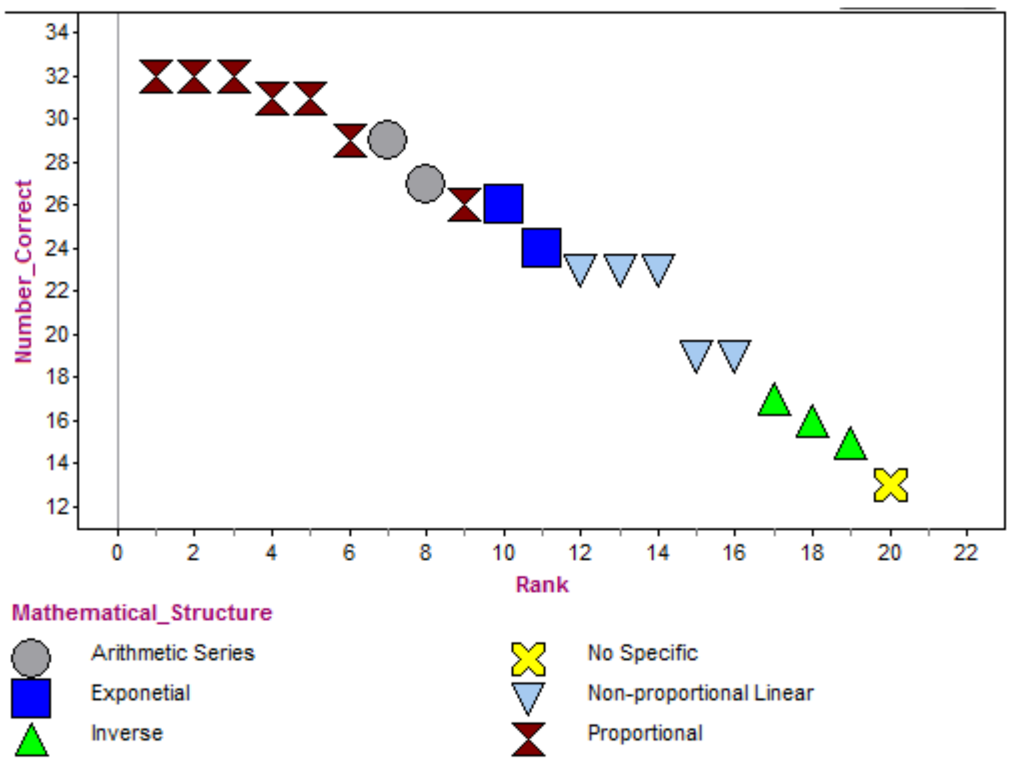

Figure 2. Structure of appropriateness items with number of correct responses.

Interestingly, nearly one-third of the teachers misidentified non-proportional linear situations as proportional. Linear functions are a significant and heavily emphasized topic in school mathematics and therefore are important for teachers to be able to differentiate from the more specific case of proportional situations. This finding led us to further investigate the knowledge resources these teachers used when evaluating the appropriateness of proportional reasoning in a linear situation, specifically the Thermometers task.

Question 4: Resources Used in Determining Whether a Situation Involved Proportion

Preliminary results from analysis of reasoning by thirteen of the participants on the non-proportional Thermometers task are shared here. Eight participants (Group 1 - Heather, Eileen, Ella, Matt, Alan, Magen, Larissa, and Tonya) correctly identified the situation as non-proportional. Three participants (Group 2 - Tori, Kathleen, and Allison), first identified the situation as proportional but changed their mind during the interview to identify the situation as non-proportional. In fact, Tori changed her mind twice and ended by identifying the situation as proportional. Two participants (Group 3 - David and Bridgett) identified the situation as proportional. All names are pseudonyms.

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Proportional Knowledge Resources. Many participants in Groups 1 and 2 used rules, scaling up and/or down, and equivalence to appropriately identify this Thermometers task as non-proportional. All the participants in Group 1 were able to provide a clear *rule* that stated the generalizable relationship between the two thermometers. For instance, Ella claimed that "the blue equals the red plus two". On the other hand, the participants from Group 2 and Bridgett (Group 3) did not provide a rule even when they were asked explicitly to do so. David (Group 3) was able to provide a rule, however he incorrectly identified the relationship as a proportional one. Creating a generalizable rule was a knowledge resource that participants in Group 1 used when they correctly identified the situation as non-proportional from the beginning.

Three participants from Group 1 (Tonya, Larissa and Meagan) used the idea of *scaling up/down* to explain why the relationship between the two thermometers was not proportional. For example, Larissa stated, "if the red was at one and blue was at two and it was a times two and then two and then four, then yes, it would be proportional. But not in this case." Allison (Group 2) also used this knowledge resource to resolve the issue she had with the difference between the thermometers being "always just two". At first she claimed, "it's proportionally it's going up the same when you drag it". However, as she continued to move the thermometers, she said, "If they were similar it wouldn't always be two because if something's four and two, if I double it to eight, that would be four if they were proportionally the same. And that's not happening here." She used multiplication to determine the equivalent ratios that would be found in a proportional relationship (*Scaling Up/Down*), and then determined that the relationship was non-proportional.

Two participants from Group 2 (Tonya and Larissa) explained the situation as non-proportional by using the idea of *equivalence* "because it's add two, there's no equivalent; the fractions created wouldn't be equivalent" (Larissa). Surprisingly, David (Group 3), who incorrectly identified the relationship as proportional, shared an accurate definition of proportions but said he was "having a hard time putting that onto this [the thermometers situation]".

Additive Knowledge Resources. Consistent with previous research on the use of language that suggests additive reasoning (e.g., Lamon, 2007; Nagar, Weiland, Orrill, & Burke, 2015), nine out of thirteen participants (at least one from each group) drew on the idea of *One Unit at a Time*. These participants claimed that the thermometers move "up one unit at a time" (Heather, Group 1). The participants used this resource to determine whether the situation was proportional and/or to explain why the situation is not proportional. For instance, Eileen (Group 1) said, "both of the bars…are moving by one unit amount…which means they are not moving in proportion to each other." Tori (Group 2) at first determined that the situation is proportional, but then she explained that when she thinks "of that [the relation between the bars] as a fraction, three fifths" and then dragged "it to where the blue is at eight, the red is at six. That's not proportional". Interestingly, she continued to explore the relationship and found that there is a constant relationship where "for every increment for red, there's an increase of one for the blue" and determined it to be a proportion. Both Bridgett and David (Group 3) also used this knowledge resource. Like Tori, David found the idea of *One Unit at a Time* as related to proportion when he claimed that "talking about the slope, the rate, it is proportional, they're going up one unit". Bridgett did not mention this resource in the context of proportion.

Conclusions

Determining whether a given situation was proportional or not was most challenging for our participants when the situation was non-proportional. Interestingly no attributes of the teachers' background seemed to relate to their ability to identify a proportional situation. Participants were able to successfully use *Rule*, *Scaling*, and *Equivalence* to identify non-proportional situations. The use of additive reasoning was common across participants (nine out of thirteen) regardless of their ability to correctly identify the situation as proportion or not. Since proportional reasoning is multiplicative

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this could suggest that teachers have the same tendency to rely on build-up strategies as their students (Lamon, 2007).

Surprisingly, we also did not observe participants relying on comparisons of quantities to determine whether the relationship was proportional. We find this interesting because the basic definition of a ratio is a multiplicative comparison of two quantities, something the participants did not draw upon in their reasoning. Instead, most of the participants referred to ratios but used additive language as they described their thinking. This suggests that teachers may not draw upon their understanding of ratio as comparison when they identify proportional situations. We noted that they did not seem to clarify whether the comparison was additive or multiplicative nor did they appear to rely on the definition of ratio which would have suggested comparing quantities rather than build-up strategies (Lobato, Ellis, Charles & Zbiek, 2010).

Our research suggests two main findings. First, teachers seem to understand their own mathematical abilities. This is important because it contradicts the widespread warnings about the suspect nature of self-reported data. At a large grainsize, these teachers had a relatively accurate assessment of their understandings. Second, there may be knowledge resources that are more useful for determining whether a situation is proportional. The teachers in this study had greater success with *Rule*, *Scaling*, and *Equivalence* than with other resources that were tried. These findings suggest that professional developers could rely more on teachers to provide insights into their own needs in content knowledge development. It also suggests that teacher development should potentially include explicit discussion of the use of different approaches to reason about the proportionality of a situation. Future research should also explore if these participants use of knowledge resources is representative of middle grades teachers.

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References

Akar, G. (2010). Different levels of reasoning in within state ratio conception and the conceptualization of rate: A possible example. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.). *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 4., pp.711–719). Columbus, OH: The University of Ohio.

Bédard, J., & Chi, M. T. H. (1992). Expertise. *Current Directions in Psychological Science*, *1*(4), 135-139.

Corbin, J., & Strauss, A. (2007). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Thousand Oaks, CA: Sage Publications.

diSessa, A. A. (1988). Knowledge in pieces. In G. Forman & P. Pufall (Eds.), *Constructivism in the computer age* (pp. 49-70). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

diSessa, A. A. (2006). A history of conceptual change research: Threads and fault lines. In R. K. Sawyer (Ed.). *The Cambridge handbook of the learning sciences* (pp. 265-282). New York: Cambridge University Press.

Field, A. (2013). *Discovering statistics using IBM SPSS Statistics* (4th ed.). London: Sage Publications Ltd.

Harel, G., & Behr, M. (1995) Teachers' solutions for multiplicative problems. *Hiroshima Journal of Mathematics Education, 3*, 31-51.

Harel, G., Behr, M., Post, T., & Lesh, R. (1992). The blocks task: Comparative analysis of the task with other proportion tasks and qualitative reasoning skills of seventh-grade children in solving the task. *Cognition and Instruction, 9*(1), 45-06.

Lobato, J., Ellis, A. B., Charles R. I. & Zbiek, R. M. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics: Grades 6-8*. Essential understanding series. Reston, VA: National Council of Teachers of Mathematics.

Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 629–667). Reston, VA: National Council of Teachers of Mathematics.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Learning Mathematics for Teaching (2007). *Survey of teachers of mathematics: Form LMT PR-2007.* Ann Arbor, MI: University of Michigan.
Ma, L. (1999). *Knowing* a*nd teaching elementary mathematics: Teacher's understanding of fundamental mathematics in China and the United States*. Mahwah: Laurence Erlbaum Associates.
Nagar, G. G., Weiland, T., Orrill, C. H., & Burke, J. P. (2015). Teachers' understanding of ratios and their connections to fractions. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 764-771). East Lansing, MI: Michigan State University.
National Governors Association & Council of Chief State School Officers [CCSSO]. (2010). *Common core state standards for mathematics*. Washington DC: Authors. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf
Orrill, C. H., Izsák, A, Cohen, A., Templin, J., & Lobato, J. (2010). Preliminary observations of teachers' multiplicative reasoning: Insights from Does it Work and Diagnosing Teachers' Multiplicative Reasoning projects. Technical Report #6. Dartmouth, MA: Kaput Center for Research and Innovation in STEM Education, University of Massachusetts.
Post, T., Harel, G., Behr, M., & Lesh, R. (1988). Intermediate teachers knowledge of rational number concepts. In Fennema et al. (Eds.), *Papers from First Wisconsin Symposium for Research on Teaching and Learning Mathematics* (pp. 194–219). Madison, WI: Wisconsin Center for Education Research.
Riley, K. R. (2010). Teachers' understanding of proportional reasoning. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1055–1061). Columbus, OH: The Ohio State University.
Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meaning. Journal of Mathematics Teacher Education, 10(4-6), 415-432.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.