FROM PROBLEM SOLVING TO LEARNING THEORIES: UNPACKING A THREE-STAGE PROGRESSION OF UNIT AREA ITERATION

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Based on the findings of a larger study on middle school students' problem solving behaviors, we identified three sub-components for understanding the iteration of unit area in this paper: the mechanism of iteration, an identical unit, and the specificity of the identical unit. A three-stage progression on the acquisition of these components is elaborated through the analysis of the students' work.

Keywords: Learning Trajectories (or Progressions), Geometry and Geometrical and Spatial Thinking, Problem Solving

Background

The research community's knowledge on how individuals learn/understand mathematical concepts serves as an important resource when investigating problem solving processes. The findings of mathematical problem solving studies, in return, could serve as a foundational resource for developing learning theories. In our previous study on middle school students' problem solving behaviors, which was designed to reveal students' ways of knowing and thinking by unpacking the relationship among mathematical concepts, cognitive behaviors, and metacognitive behaviors (Zhang, 2010), we proposed a concept development framework for areas based on Vygotsky's (1962) concept formation theory, Berger's (2004) appropriation theory, and Battista's (2012) Cognition-Based Assessment (CBA) levels. The framework served as a platform for designing and analyzing problem solving interviews. One of the findings of the study suggested that the concept development framework of areas could be further refined to characterize sub-components to be more explicit as a guideline for teaching and studying this particular concept. This paper elaborates on a three-stage progression of the iteration of unit area based on the findings.

Theoretical Framework

Vygotsky's concept formation theory and Berger's appropriation theory were used as the structure of the proposed framework, while Battista's CBA levels served as a key reference for the specific stages along with the corresponding cognitive behaviors in the framework.

Vygotsky's theory proposes a framework for an individual's concept development within a social environment, while Berger's theory proposes an interpretation of Vygotsky's theory in the domain of mathematics by adjusting certain stages. Both theories break down any concept development into three phases: heap, complex, and concept. In the heap phase, the learner associates a sign with another because of physical context or circumstance instead of any inherent or mathematical property of the signs. In the complex phase, objects are united in an individual's mind not only by his or her impressions, but also by concrete and factual bonds between them. In the concept phase, the bonds between objects are abstract and logical.

The formation stages for the target concept of study (the concept of area), guided by the two theories described above, are illustrated in Figure 1. The developmental stages were refined based on the pretest responses from 44 middle school students and guided the selection of the interview tasks as well as the analysis of the relationship among mathematical concepts, cognitive behaviors, and metacognitive behaviors emerged from the interview results.

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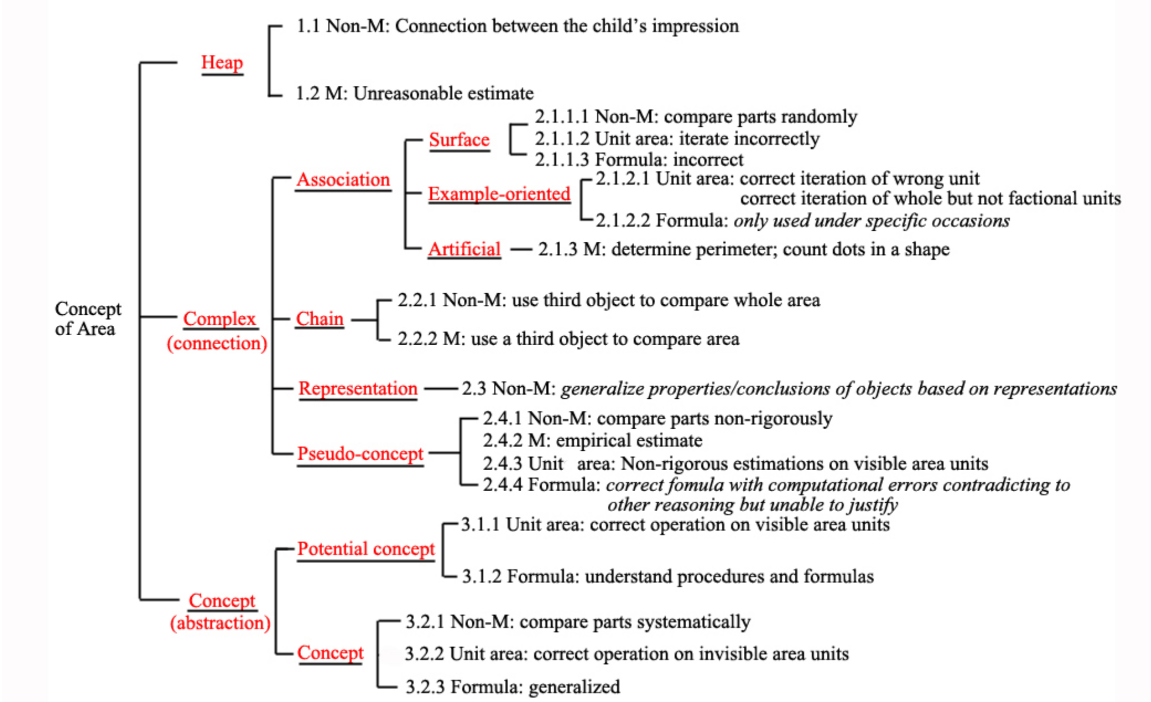

Figure 1. Developmental stages of the concept of area.

In the concept framework, there are three major components under the concept of area: non-measurement reasoning, unit area, and formula; this paper focuses on the unit area component. The specific stages involved in the progression of unit area iteration include: 2.1.1.2. Surface Association Complex – Unit area (iterate incorrectly), 2.1.2.1. Example-oriented Association Complex – Unit area (correct iteration of wrong unit and correct iteration of whole but not fractional units), 3.1.1. Potential concept – Unit area (correct operation on visible area units), and 3.2.2. Concept – Unit area (correct operation on invisible area units).

Methods

Participants

Five individuals from a population of 44 sixth grade students were selected to participate in interviews. A pretest was administered to the 44 students and each individual's developmental status revealed in the responses was categorized as "overall low" (all responses were rated as Heap and non-Pseudo-concept Complex stages), "varied" (responses were rated across Heap to Concept stages), and "overall high" (responses were rated as Pseudo-concept Complex and Concept stages). Among the five participants, one exhibited a low status, one exhibited a high status, and three exhibited varied statuses.

Instrument and Data collection

The five participants were interviewed individually. Each interview consisted of two parts. During the background interview part, the participants' mathematics background information, their beliefs about mathematics, and their views on the value of mathematics for their lives were elicited. During the second part, problem solving interviews, the participants worked on specific mathematical tasks, while interviewer interventions were limited to eliciting clarifications, explanations, or justifications when needed.

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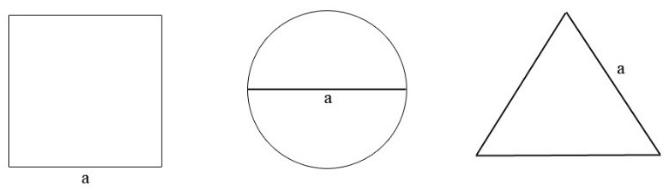
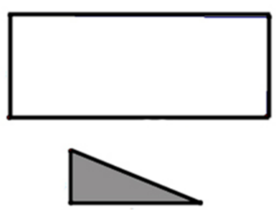
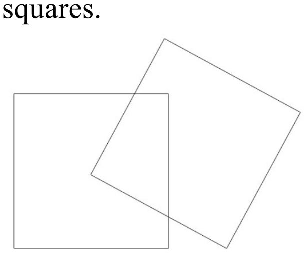
Five problems were used during the interviews. All problems were related to the concept of area and allowed the participants to tackle the tasks from different stages of concept development. The problems were designed to potentially cover a wide range of concept stages. The process and rationale of instrument design was reported in a previous paper (Zhang, Manouchehri, & Tague, 2015).

The students' working examples referred in this paper are from three out of the five interview questions, which are illustrated in Table 1.

Table 1: Interview problems

Compare areas problem Which of the regions shown below has the largest area? How would you order them? 
Shaded Triangle problem How many of the shaded triangles shown below are needed to exactly cover the surface of the rectangle? Please explain your answer. 
Intersected Area problem Two squares, each *s* on a side, are placed such that the corner on one square lies on the center of the other. Describe, in terms of *s*, the range of possible areas representing the intersections of the two squares. 

In the Shaded triangle problem, the measurement of the rectangle and the triangle was deliberately removed to test how the participants may solve the problem under this condition. The participants had been expected to determine or question the relationship (2:1 ratio) between the measures of the rectangle and the triangle.

Data analysis

Data analysis consisted of three aspects: concept stages, cognitive behaviors, and metacognitive behaviors. First, each participant's key cognitive behaviors during each problem solving episode were documented. Second, a summary of observed concept stages and metacognitive behaviors during the episode were catalogued and noted. Finally, a cross analysis of the observed concept stages, metacognitive behaviors, and the relationship between them concluded the analysis phase. This process was followed for each of the five tasks used.

Results

One of the findings indicated that the progression of the iteration of unit area included three components: *the mechanism of iteration, an identical unit,* and *the specificity of the identical unit*. The data suggested a specific order of stages for the acquisition of these components.

Stage 1 – understanding the mechanism of iteration

When an individual understands the *mechanism of iteration* yet is not able to visualize a given shape as the unit, s/he is at 2.1.1.2. Surface Association Complex – Unit area stage and would iterate different units (e.g. triangles of different shapes and sizes).

The findings suggested that an individual could have different visualization abilities for different shapes. For example, when solving the Shaded Triangle problem, a participant who was not able to visualize the given triangle as a unit, iterated 10 triangles with varied shapes and sizes to cover the entire rectangle (Figure 2). Being asked whether she could use a different approach to solve the problem, she rotated the triangle and formed a small rectangle as a unit, then she was able to correctly iterate the small rectangle as a unit to cover the entire rectangle (Figure 3).

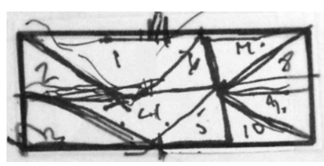


Figure 2. Incorrect visualization of a triangle as a unit.

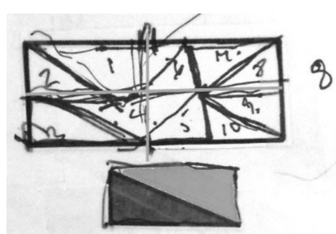


Figure 3. Correct visualization of a rectangle as a unit.

Stage 2 – understanding the mechanism of iteration and an identical unit

When an individual understands the *mechanism of iteration* and the *identical unit*, s/he is at 2.1.2.1. Example-oriented Association Complex – Unit area stage and could correctly iterate an identical (but not the given) unit.

When solving the Shaded Triangle problem, a participant who correctly iterated eight triangles in the rectangle (Figure 4) claimed that one could iterate differently as long as the total number of triangles was eight (as in Figure 5).

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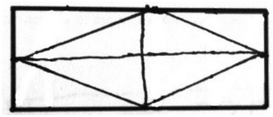

Figure 4. An individual's correct visualization of unit rectangles.

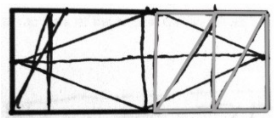

Figure 5. The same individual's Example-oriented Association Complex – Unit area reasoning.

This participant was not aware of the specificity of the identical unit, but he was able to justify the validity of the answer "eight triangles" and abandoned the numerical answer obtained by the correct formula which was 7.57 (the number was incorrect due to inaccurate measures). His Example-oriented Association Complex – Unit area level of understanding was not revealed until the end of the interview. It was triggered by the prompt of "can there be two answers to a problem?" which was intended to elicit his evaluation on the two different answers obtained from visual and formulaic approaches.

Stage 3 – understanding the mechanism of iteration, an identical unit, and specificity of the identical unit

When an individual understands the *mechanism of iteration*, an *identical unit*, and the *specificity of the identical unit*, s/he is at 3.1.1. Potential Concept – Unit area stage or 3.2.2. Concept – Unit area stage.

At the Potential Concept – Unit area stage, an individual is not required to (flexibly) define the area unit prior to the processes of decomposing (e.g. assigning proportional relation between partial squares and whole squares) and reconstructing (e.g. converting partial squares to whole squares) but relying on standard visible area units. For example, when solving the Compare Areas problem, a participant drew unit squares on the circle to find its area (Figure 6) since he forgot the area formula for circles.

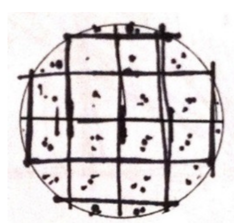

Figure 6. A Potential Concept – Unit area reasoning.

While at the Concept – Unit area stage, an individual takes full control over the area units in terms of their shapes, sizes, orientations, and other properties during the restructuring. For example, when solving the Intersected Area problem, the participant defined the overlapping area (a quadrilateral) as her unit, iterated it three times to cover the entire square (Figure 7), and reached the conclusion that the intersected area is a quarter of the whole square.

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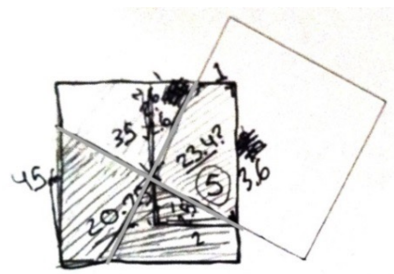

Figure 7. A Concept – Unit area reasoning.

Conclusion

Among the three components under the concept of area (non-measurement reasoning, unit area, and formula), we identified three sub-components for understanding the iteration of unit area: the mechanism of iteration, an identical unit, and the specificity of the identical unit. A specific order of the acquisition of these components was outlined through the analysis of the participants' work as the three stages in the previous section. Each stage corresponds to one or two stages in the concept framework.

Comparing to the four stages in the concept framework (Figure 1) and the progression of area-unit iteration outlined in CBA levels (Table 2), the proposed three-stage progression provides a more explicit conceptual breakdown of the iteration of unit area. In this way, more accurate assessments of students' knowledge can be developed and used, both in research and in the classroom.

Table 2: CBA levels for unit area iteration

Level	Sub-level	Description
M0		Student uses numbers in ways unconnected to appropriate area-unit iteration.
M1		Student incorrectly iterates area-units.
	M1.1	Student iterates single area units incorrectly.
	M1.2	Student decomposes shapes into parts incorrectly.
	M1.3	Student iterates area-units incorrectly, but eliminates double-counting errors.
M2		Student correctly iterates all area units one by one.
	M2.1	Student correctly iterates whole units, but not fractional units.
	M2.2	Student correctly iterates whole units and simple fractional units.
M3		Student correctly operates on composites of visible area-units.

Both concept development theories and learning progressions emphasize the non-hierarchical nature of their stages and levels, i.e., individuals may jump around the stages/levels without

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following a specific order (Smith et al., 2006). Whether the three sub-components can be acquired in different orders remains to be examined by future studies.

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